Deformation Theory and Rational Homotopy Type

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November 8, 2012

Abstract

We regard the classification of rational homotopy types as a problem in algebraic deformation theory: any space with given cohomology is a perturbation, or deformation, of the "formal" space with that cohomology. The classifying space is then a "moduli" space — a certain quotient of an algebraic variety of perturbations. The description we give of this moduli space links it with corresponding structures in homotopy theory, especially the classification of fibres spaces $F \to E \xrightarrow{p} B$ with fixed fibre F in terms of homotopy classes of maps of the base B into a classifying space constructed from $Aut(F)$, the monoid of homotopy equivalences of F to itself. We adopt the philosophy, later promoted by Deligne in response to Goldman and Millson, that any problem in deformation theory is "controlled" by a differential graded Lie algebra, unique up to homology equivalence (quasi-isomorphism) of dg Lie algebras. Here we extend this philosophy further to control by L_∞ -algebras.

In memory of Dan Quillen who established the foundation on which this work rests.

Contents

1 Introduction

In this paper, we regard the classification of rational homotopy types as a problem in algebraic deformation theory: any space with given cohomology is a perturbation, or deformation, of the "formal" space with that cohomology. The classifying space is then a "moduli" space — a certain quotient of an algebraic variety of perturbations. The description we give of this moduli space links it with others which occur in algebra and topology, for example, the moduli spaces of algebras or complex manifolds. On the other hand, our dual vision emphasizes the analogy with corresponding structures in homotopy theory, especially the classification of fibres spaces $F \to E \xrightarrow{p} B$ with fixed fibre F in terms of homotopy classes of maps of the base B into a classifying space constructed from $Aut(F)$, the monoid of homotopy equivalences of F to itself. In particular, the moduli space of rational homotopy types with fixed cohomology algebra can be identified with the space of "path components" of a certain differential graded coalgebra.

Although the majority of this paper is concerned with constructing and verifying the relevant machinery, the final sections are devoted to a variety of examples, which should be accessible without much of the machinery and might provide motivation for reading the technical details of the earlier sections.

Portions of our work first appeared in print in [56, 57] and then in 'samizdat' versions over the intervening decades (!), partly due to some consequences of the mixture of languages. Some of those versions have worked there way into work of other researchers; we have tried to maintain much of the flavor of our early work while taking advantage of progress made by others.

Crucially, throughout this paper, the ground field is the rational numbers, $\mathbb{Q}$ (characteristic 0 is really the relevant algebraic fact), although parts of it make sense even over the integers.

1.1 Background

Rational homotopy theory regards **rational homotopy equivalence** of two simply connected spaces as the equivalence relation generated by the existence of a map $f : X \to Y$ inducing an isomorphism $f^* : H^*(Y; \mathbf{Q}) \to H^*(X, \mathbf{Q})$. Here we are much closer to a complete classification than in the ordinary (integral) homotopy category. An obvious invariant is the cohomology algebra $H^*(X; \mathbf{Q})$. Halperin and Stasheff [20] showed that all simply connected spaces X with fixed cohomology algebra $\mathcal{H}$ of finite type over $\mathbf{Q}$ can be described (up to rational homotopy type) as follows: (*Henceforth 'space' shall mean 'simply connected space of finite type' unless otherwise specified.*)

Resolve $\mathcal{H}$ by a **d**(ifferential) **g**(raded) **c**(ommutative) **a**(lgebra) (SZ, d) which is connected and free as a graded commutative algebra with a map $(SZ, d) \to \mathcal{H}$ of dgcas inducing $H(SZ, d) \simeq \mathcal{H}$. Here S denotes graded symmetric algebra. (See section 2.1 for details, especially in re: the various gradings involved.) The notation Λ instead of S is often used within rational homotopy theory, where it is a historical accident derived from de Rham theory.

Let $A^*(X)$ denote a differential graded commutative algebra of "differential forms over the rationals" for the space X, e.g. Sullivan's version of the deRham complex [63, 6]). Given an isomorphism $i : \mathcal{H} \xrightarrow{\cong} H^*(X)$, there is a perturbation p (a derivation of SZ of degree 1 which lowers resolution degree by at least 2 such that $(d+p)^2 = 0$) and a map of dga's $(SZ, d+p) \to A^*(X)$ inducing an isomorphism of rational cohomology. If X and Y have the same rational homotopy type, the perturbations p_X and p_Y must be related in a certain way, spelled out in §2.2. This is one of several ways (cf. [36, 13]) it can be seen that

Main Theorem 1.1. *For fixed $\mathcal{H}$, the set of homotopy types of pairs $(X, i : \mathcal{H} \simeq H(X))$ can be represented as the quotient V/G of a (perhaps infinite dimensional) conical rational algebraic variety V modulo a pro-unipotent (algebraic) group action G.*

Corollary 1.2. *The set of rational homotopy types with fixed cohomology $\mathcal{H}$ can be represented as a quotient $Aut\,\mathcal{H}\backslash V/G$.*

1.2 Control by DGLAs

The variety and the group can be expressed in the following terms: let $Der\ SZ$ denote the graded Lie algebra of graded derivations of SZ, which is itself a dg Lie algebra (see Definition 3.1) with the commutator bracket and the differential induced by the internal differential on SZ. Let $L \subset Der\,SZ$ be the sub–Lie algebra of derivations that decrease the sum of the total degree plus the resolution degree. The variety $V \subset L^1$ is precisely $V = \{p \in L^1 | (d+p)^2 = 0\}$. In fact, V is the cone on a projective variety (of possibly infinite dimension) §2.3 The pro-unipotent group G is $\exp L$, which acts via the adjoint action of L on $d + p$.

We said above that we regard our problem as one of deformation theory in the (homological) algebra sense. A commutative algebra $\mathcal{H}$ has a Tate resolution which is an **almost free** commutative dga SZ, that is, free as graded commutative algebra, ignoring the differential. A deformation of $\mathcal{H}$ corresponds to a change in differential $d \to d + p$ on SZ. Instead of $L \subset Der\,SZ$ as above, the sub-dg Lie algebra $\bar{L} \subset Der\,SZ$ of nonpositive resolution degree is used.

As far as we know, deformation theory arose with work on families of complex structures. Early on, these were expressed in terms of a moduli space [53, 69]. This began with Riemann, who first introduced the term "moduli". He proved that the number of moduli of a surface of genus 0 was 0. 1 for genus 1 and $3g - 3$ for $g > 1$. These are the same as the numbers of quadratic differentials on the surface, which was perhaps the impetus for Teichmueller's work identifying these as "infinitesimal deformations". Teichmueller was probably the first to introduce the concept of an infinitesimal deformation of a complex manifold, but in his form it was restricted to Riemann surfaces and could not be extended to higher dimensions. Froelicher and Nijenhuis [14] gave the appropriate definition of an infinitesimal deformation of complex structure of arbitrary dimension. This was the essential starting point of the work by Kodaira and Spencer [31] and with Nirenberg [30]. in terms of deformations. To vary complex structure, one varied the differential in the Dolbeault complex of a complex

manifold. In this context, analytic questions were important, especially for convergence of power series solutions.(See both Doran's and Mazur's historical annotated bibliographies [9, 41] for richer details.)

Algebraic deformation theory began with Gerstenhaber's seminal paper [15]:

> This paper contains the definitions and certain elementary theorems of a deformation theory for rings and algebras....mainly associative rings and algebras with only brief allusions to the Lie case, but the definitions hold for wider classes of algebras.

Subsequent work of Nijenhuis and Richardson [49] provided more detail for the Lie case. Computations were in terms of formal power series. If a formal deformation solution was found, its convergence could be studied, sometimes without estimates. This is the context in which we work in studying deformations of rational homotopy types. Unfortunately, the word *'formal'* appears here in another context, referring to a dgca which is weakly equivalent to its homology as a dgca.

An essential ingredient of our work is the combination of this algebraic geometric aspect with a homotopy point of view. Indeed we adopted the philosophy, later promoted by Deligne [8] in response to Goldman and Millson [17] (see [18] for a history of that development), that any problem in deformation theory is "controlled" by a differential graded Lie algebra, unique up to homology equivalence (quasi-isomorphism or quism) (cf. 6.1) of dg Lie algebras . In Section 6, we extend this philosophy further to control by L_∞ -algebras.

For a problem controlled by a general dg Lie algebra, the deformation equation (also known as the Master Equation in the physics and physics inspired literature and now most commonly as the Maurer-Cartan equation) is written

$$dp + 1/2[p,p] = 0.$$

It appears in this form (though with the opposite sign and without the 1/2, both of which are irrelevant) in the early works on deformation of complex structure by Kodaira, Nirenberg and Spencer [30] and Kuranishi [33].

Differential graded Lie algebras provide a natural setting in which to pursue the obstruction method for trying to integrate "infinitesimal deformations", elements of $H^1(DerSZ)$, to full deformations. In that regard, $H^*(DerSZ)$ appears not only as a graded Lie algebra (in the obvious way) but also as a strong homotopy Lie algebra, a concept proven to be of significance in physics, especially in closed string field theory[62, 75, 74]. This allows us to go beyond the consideration of quadratic varieties so prominent in Goldman and Millson.

Since we are trying to describe the space of homotopy types, it is natural to do so in homotopy theoretic terms. Quillen's approach to rational homotopy theory emphasizes differential graded Lie algebras in another way. The rational homotopy groups $\pi_*(\Omega X) \otimes \mathbf{Q}$ form a graded Lie algebra under Samelson product. Moreover, Quillen [51] produces a non–trivial differential graded Lie algebra λ_X which not only gives $\pi_*(\Omega X) \otimes \mathbf{Q}$ as $H(\lambda_X)$ but also faithfully records the rational homotopy type of X. A simplistic way of characterizing such an λ_X for nice X is as follows: There is a standard construction A such that for any dg

Lie algebra L, we have $A(L)$ as a dgca, and for λ_X, we have $A(\lambda_X) \to A^*(X)$ as a model for X [39]. (For ordinary Lie algebras L, $A(L)$ is the standard (Chevalley-Eilenberg) complex of alternating forms used to define Lie algebra cohomology [7].)

On the other hand, A can also be applied to the dg Lie algebra $L \subset Der\, SZ$ above; then it plays the role of a classifying space.

Main Homotopy Theorem 1.3. *For a simply connected graded commutative algebra of finite type $\mathcal{H}$, homotopy types of pairs $(X, i : \mathcal{H} \simeq H(X))$ are in 1–1 correspondence with homotopy classes of dga maps $A(L) \to \mathbf{Q}$.*

Now $A(L)$ will in general *not* be connected, so the homotopy types correspond to the "path components". One advantage of this approach is that it suggests the homotopy invariance of the classifying objects. Indeed, if (SZ, d) and (SZ', d) are homotopy equivalent as dga's, then $Der\, SZ$ and $Der\, SZ'$ will be weakly homotopy equivalent as dg Lie algebras) (Theorem 3.9). In particular, if (SZ, d) is a resolution of $\mathcal{H}$, then $H(Der\,(SZ, d))$ will be an invariant of $\mathcal{H}$, not just as a graded Lie algebra but, in fact, as an L_∞-algebra. Similarly if L and L' are homotopy equivalent in the category of dg Lie algebras or L_∞-algebras, then $A(L)$ and $A(L')$ will be homotopy equivalent dgca's.

The variety and group of the Main Theorem depend on a particular choice of L, but the moduli space V/G depends only on the "homology type of L (in degree 1)"(Definition 5.4).

We show in **??** that for a simply connected dgca A with finite dimensional homology, the entire dg Lie algebra $L = Der\, A$ is homology equivalent to a dg Lie algebra K which is finite dimensional in each degree. In particular, the V and G of the classification problem may be taken to be finite dimensional when $\mathcal{H}$ has finite dimension.

The varieties V which occur above are the "versal deformation" spaces of algebraic geometry. Our combination of the algebraic–geometric and homotopy points of view leads to a particularly useful (and conceptually significant) representation of the minimal versal deformation space called "miniversal". The coordinate ring of the miniversal W is approximated by the 0^{th} cohomology of $A(L)$, and its Zariski tangent space (the vector space of infinitesimal deformations of structure) is $H^1(L)$. However, the quotient action on the miniversal W is no longer given by a group action in general, but only by a 'foliation' (in the generalized sense of e.g. Haefliger) induced from an L_∞-action of $H^0(L)$.

From this and deformation theory, we deduce information about the structure of a general $V_L/\exp L$, for example:

$V_L/\exp L$ is one point provided $H^1(L) = 0$;

$V_L/\exp L$ is a variety provided $H^0(L) = 0$; and

V_L is non-singular (a flat affine space) provided $H^2(L) = 0$.

If L is formal, i.e., if there is a dg Lie algebra map $L \to H(L)$ inducing a homology isomorphism, then V_L is isomorphic to the quadratic variety $\{\theta \in H^1(L) | [\theta, \theta] = 0\}$. If L is not formal, then $[\theta, \theta]$ may be only the *primary obstruction.* Let V_1 denote $\{\theta \in H^1(L) | [\theta, \theta] = 0\}$. There is a *secondary* obstruction defined on V_1 which vanishes on a subspace denoted V_2. The secondary obstruction was first addressed in specific examples by Douady [10]. Proceeding in this way, the succesive obstructions to constructing a full

perturbation can be described in terms of Lie-Massey brackets [52] and lead to a filtration $H^1(L) \supset V_1 \supset V_2 \ldots$ with intersection being V_L.

There are conceptual and computational advantages to replacing (SZ, d) by the quadratic model $A(\lambda_X)$ where X is the formal space (determined by just the cohomology algebra $\mathcal{H}$) or better yet by $A(L(\mathcal{H}))$ where $L(\mathcal{H})$ is the free Lie coalgebra model in which the differential is generated by the multiplication $\mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$ (§3). (In the ungraded case, $L(\mathcal{H})$ is the complex for the Harrison [22] cohomology of a commutative algebra and, here in characteristic 0, also known as the cotangent complex.) While (SZ, d) corresponds to a Postnikov system, $L(\mathcal{H})$ relates to cellular data for the formal space determined by $\mathcal{H}$. A perturbation of $L(\mathcal{H})$ which decreases bracket length can be identified with a Lie symmetric map $\mathcal{H}^{\otimes k} \to \mathcal{H}$ of degree $2 - k$; this can sometimes be usefully interpreted via Massey products.

1.3 Applications

Although the general theory we develop has some intrinsic interest, it is in the application to specific rational homtopy types that some readers will find the appropriate justification. In the last two sections, we study several families of examples. Finding a complete set of invariants for rational homotopy types seems to be of about the same order of difficulty as finding a complete set of invariants for the G–orbits of a variety V, but, for special cohomology rings, a lot more can be said [20, 13, 37, 47]; for example, Massey product structures can be very helpful, though they are in general described in a form that is unsatisfactory. When the dimension is a small multiple of the connectivity, results can be given in a neat form (§8). The deformations of the homotopy type of a bouquet of spheres allows for detailed computations and reveals some pleasant surprises.

We have mentioned that $A(L)$ behaves like a classifying space. We carry this insight to fruition in §9 by considering nilpotent topological fibrations

$$F \to E \to B$$

of nice spaces. For these, Sullivan ([63] p. 313) has proposed an algebraic model for $B\,Aut\,F$, the classifying space for nilpotent fibrations with fibre F. His model is essentially $A\,(Der\,SZ)$ suitably altered to model a nice space where SZ models F. Here we view a fibration as a "deformation" of the trivial fibration $F \times B$. The set of such fibrations is a quotient $V_L/\exp L$ of the type considered above. (L is the complete tensor product of $Der\,SZ$ with an algebraic model of the base, cf. 9.4.) It then follows that $B\,Aut\,F$ is modeled by $A(D)$ for a suitable sub dg Lie algebra D of $Der\,SZ$, in the sense that equivalence classes of fibrations $F \to E \to B$ correspond to homotopy classes of maps of $A(D)$ into the model of B, at least if B is simply connected (9.6). By placing suitable restrictions on D, we obtain certain special kinds of fibrations and also "almost" a space which simultaneously classifies perturbations of F and F fibrations. Moreover, we can describe, in terms of the weight of the perturbation, which fibrations correspond to perturbing the homotopy type of $F \times B$ or the algebra structure of $H(F \times B)$. The principal problem left is to decide whether an arbitrary quotient variety can arise as a moduli space for homotopy types or fibrations.

1.4 Outline

We have tried to write this paper so it will be of interest and accessible to algebraic topologists, algebraic geometers and algebraists. We have presented at least a quick sketch of all the machinery we use. It would not hurt to be somewhat familiar with rational homotopy theory (especially [63, 19, 20, 3, 64]) and Sullivan's models in particular. The examples 4.3, 6.4, 6.5, and 6.6 in [20] may help put the abstract constructions in focus. In §7.4, we move from homotopy theoretic language to that of algebraic geometry for a more traditional approach to moduli functors.

In §2 , we recall the notion of a dgca model of a rational homotopy type and particularly the formal one given by the Tate–Jozefiak resolution, as well as its perturbations. We show the perturbations form a cone on a projective variety.

In §3, we recall the standard constructions $C(L)$ and $A(L)$ for a dg Lie algebra L, as well as the adjoint $L(C)$ from dgcc's to dg Lie algebras . The homology or cohomology of these constructions has significance in both homotopy and Lie algebra theory, but we are more concerned with the constructions themselves. Since some of our results depend on finite type or boundedness restrictions, we show how the construction $C(L)$ does not change homotopy type under certain changes in L. We also look at the special model $A(L(\mathcal{H}))$ and at $L(\mathcal{H})$ itself, where $\mathcal{H} = H(C)$.

In §4, we look at the key notion of homotopy of dg algebraic maps (as opposed to homotopy of chain maps) and the corresponding notion for coalgebras. The corresponding relation between perturbations is best expressed via a differential equation. In §4.2, we give a direct proof for the invariance of $V_L/\text{exp}L^0$ in terms of $H(L)$. is induced on $H(L)$ In §5, we use the differential equation to complete the proof of the main results: that for the appropriate dg Lie algebra L, the set of homotopy types $(X; i : \mathcal{H} \simeq H(X))$ or equivalently the set $V_L/\text{exp}L^0$ corresponds to the set $[\mathbf{Q}, \hat{C}(L)]$ of homotopy classes of maps of $\mathbf{Q}$ into $\hat{C}(L)$, the completion of $C(L)$.

In §6, we explain how an L_∞-structure is induced on $H(L)$ so that our classification can be expressed in terms of $H(Der??)$.

In §7, we switch to the algebraic geometric language for deformation theory. We focus on the miniversal variety W_L contained in $H^1(L)$ and the corresponding moduli space.

In §8, we consider examples computationally, including relations to Massey products and examples of $\exp L^0$ actions in terms of maps of spheres.

In §9, we establish the corresponding results for fibrations and compare fibrations to perturbations of the product of the base and fibre in terms of weight conditions. We conclude with some open questions §9.4.

As the first draft of this paper was being completed (by slow convergence), we learned of the doctoral thesis of Yves Félix [12] (published as [13]) which obtains some of these results from a point of view less homotopy theoretic as far as the classification is concerned, but which focuses more on the orbit structure in V_L and is closer to classical deformation theory. Félix deforms the algebra structure as well as the higher order structure. We are happy to report our computations agree where they overlap.

Later, we learned of the thesis of Daniel Tanré and his lecture notes where he carried out

the classification in terms of Quillen models [64] (Corollaire VII.4. (4)). with slightly more restrictive hypotheses in terms of connectivity, producing the equivalent classification.

J.-C. Thomas [70] analyzes the internal structure of fibrations from a compatible but different point of view [71]. Subsequently, Berikashvili (cf. [5]) and his school have studied 'pre-differentials', which are essentially equivalence classes of our perturbations. Applications to rational homtopy types and to fibrations have been developed by Saneblidze [54] and Kadeishvili [29] respectively.

These are but some of the many results in rational homtopy theory that have appeared since the first draft of this paper, some in fact using our techniques. For an extensive bibliography, consult the one prepared by Félix [11] building on an earlier one by Bartik.

Finally, we owe the first referee a deep debt for his insistence that our results deserved better than the exposition in our first draft (compare the even more stream of consciousness preliminary version in [57]). He has forced us to gain some perspective (even pushed us toward choosing what was, at that time, the right category in which to work) which hopefully we have revealed in the presentation. In particular, the Lie algebra models which have been emphasized in [4, 47] are certainly capable of significant further utilization, whether for classifying spaces or manifolds [61]. He has also encouraged us to make extensive use of the fine expositions of Tanré [64, 67] which appeared as this manuscript began to approach the adiabatic limit. The original research in this paper was done in the late 70's and early 80's [56, 57], as will be obvious from what is *not* assumed as 'well known'. Rather than rewrite the paper in contemporary fashion, postponing it to the next millenium, we have elected to implement most of the referee's suggestions while leaving manifest the philosophy of those bygone times.

2 Models of homotopy types

We begin with a very brief summary of those features of rational homotopy theory which are relevant for our purpose.

Quillen's rational homotopy theory [51] focuses on the equivalence of the rational homotopy category of simply connected CW spaces and the homotopy category of simply connected dgcc's (differential graded commutative coalgebras) over $\mathbf{Q}$. Sullivan [63] uses dgca's (differential graded commutative algebras) and calls attention to minimal models for dgca's so as to replace homotopy equivalence by isomorphism. Halperin and Stasheff [20] discovered another class of models which turn out to be appropriate for classification and can be used without any elaborate machinery. Indeed, we recall here what little we need.

For our entire discussion, we let $\mathbf{Q}$ denote an arbitrary fixed ground field of characteristic 0. We adopt a strictly cohomological point of view, i.e. all graded vector spaces will be written with an upper index (unless otherwise noted) and all differentials on graded algebras (associative or Lie or...) will be derivations of degree 1 and square 0. As will be explicitly noted, many graded algebras we encounter will be either non–negatively (as for cochains on topological spaces) or non-positively graded (as in algebraic geometry). For an associative algebra $A = \bigoplus_{n\geq 0} A^n$ and $A^0 = \mathbf{Q}$, we call A *connected*. When needed, we will refer to **simply connected** algebras, i.e., $A^0 = \mathbf{Q}$ and $A^1 = 0$.

Definition 2.1. *Two dgca's A_1 and A_2 have the same* **rational homotopy type** *if there is a dgca A and dga maps $\phi_i : A \to A_i$ such that $\phi_i^* : H(A) \to H(A_i)$ is an isomorphism. This is an equivalence relation since such an A can always be taken to be free as a gca; it is then called a* **model** *for A_i.*

One way to construct a model is as follows:

For any connected graded commutative algebra $\mathcal{H}$ over $\mathbf{Q}$, there are free algebra resolutions $A \to \mathcal{H}$; that is, A is itself a dgca, free as gca, and the map, which is a morphism of dga's regarding $\mathcal{H}$ as having $d = 0$, induces an isomorphism $H(A) \xrightarrow{\approx} \mathcal{H}$. Indeed, there is a minimal free dgca resolution which we denote $(SZ, d) \to \mathcal{H}$ due to Jozefiak [27] which is a generalization of the Tate resolution [68] in the ungraded case.

2.1 The Tate–Jozefiak resolution in characteristic zero

The free graded commutative algebra SZ on a graded vector space Z is $E(Z^{odd}) \otimes P(Z^{even})$ where E = exterior algebra and P = polynomial algebra. In resolving a connected graded commutative algebra A by a dgca (SZ, d), the generating graded vector space Z will be bigraded:

$$Z^n = \bigoplus_{q\leq 0}^{w+q=n} Z^{q,w}.$$

We will refer to $n = w + q$ as the **total** or **topological** or **top** degree, to q as the **resolution** degree (for historical reasons), and to w as the **weight** = topological degree minus resolution

degree. When a single superscript grading appears, it will always be total degree. The graded commutativity is with respect to the total degree.

The Tate–Jozefiak resolution $(SZ, d) \to \mathcal{H}$ of a connected cga $\mathcal{H}$ has a differential d which increases total degree by 1, decreases resolution degree by 1 and hence *preserves* weight. It is a graded derivation with respect to total degree. For connected $\mathcal{H}$, we let $Q\mathcal{H} = \mathcal{H}^+/(\mathcal{H}^+ \cdot \mathcal{H}^+)$ be the module of indecomposables. The vector space $Z^{0,*}$ is $Q\mathcal{H}$. The resolution $\rho : (SZ, d) \to \mathcal{H}$ induces an isomorphism $H(\rho) : H(SZ, d) \overset{\approx}{\to} \mathcal{H}$ which identifies $\mathcal{H}$ with $H^{0,*}(SZ, d)$.

We refer to the Tate–Jozefiak resolution (SZ, d) also as the **bigraded** or **minimal model** for $\mathcal{H}$. It is minimal in the sense that the dimension of each $Z^{q,*}$ is as small as possible, but the minimality is best expressed as $dZ \subset S^+Z \cdot S^+Z$ where $S^+Z = \bigoplus\limits_{n>0} (SZ)^n$.

Several comments are in order. Just as in ordinary homological algebra, one can easily prove (SZ, d) is uniquely determined up to isomorphism. If $(SZ', d) \underset{\rho'}{\longrightarrow} \mathcal{H}$ is any other minimal free bigraded dgca with $H(\rho')$ an isomorphism, then (SZ', d) is isomorphic to (SZ, d).

2.2 The Halperin–Stasheff or filtered model

Given a dgca (A, d_A) and an isomorphism $i : \mathcal{H} \approx H(A)$, Halperin and Stasheff ([20] p. 249) construct a **perturbation** p of the Tate-Jozefiak (SZ, d) of $\mathcal{H}$ and thus a derivation $d + p$ on SZ and a map of dgca's $\pi : (SZ, d + p) \to (A, d_A)$ such that $(d + p)^2 = 0$ and

1. p decreases resolution degree by at least 2 (i.e., decreases weight, thus $(SZ, d + p)$ is filtered graded, not bigraded), and

2. $H(\pi)$ is an isomorphism $H(SZ, d + p) \approx H(A)$.

   In fact, filtering SZ by resolution degree, we have in the resulting spectral sequence

3. $(E_1, d_1) = (SZ, d)$ and $E_2 = H(SZ, d)$ which is concentrated in resolution degree 0 and,

4. by construction, $H(\pi)$ is the composite

$$H(SZ, d + p) \overset{\approx}{\to} H(SZ, d) \xrightarrow[\rho^*]{\approx} \mathcal{H} \xrightarrow[i]{\approx} H(A)$$

   with the first isomorphism being the edge morphism of the spectral sequence.

Now consider two dgca's A and B with isomorphisms

$$H(A) \underset{i}{\overset{\approx}{\leftarrow}} \mathcal{H} \underset{j}{\overset{\approx}{\rightarrow}} H(B).$$

The corresponding perturbed models $(SZ, d+p)$ and $(SZ, d+q)$ are **homotopy equivalent relative to** i **and** j (i.e., by a map ϕ such that $H(\phi) = ji^{-1}$) if and only if there exists

$(*)$ a dga map $\phi : (SZ, d + p) \to (SZ, d + q)$ such that $\phi - Id$ lowers resolution degree. (It follows that $H(\phi) = ji^{-1}$.)

Definition 2.2. *Let $(SZ, d) \to \mathcal{H}$ be the bigraded model of $\mathcal{H}$. A* **perturbation** *of d is a weight decreasing derivation p of total degree 1 such that $(d+p)^2 = 0$.*

For any perturbation, $(SZ, d+p)$ has cohomology isomorphic to the original $\mathcal{H}$. Thus the classification of homotopy types can be done in stages:

1. Fix a connected cga $\mathcal{H}$.
2. Let $V = \{$ perturbations p of the minimal model (SZ, d) for $\mathcal{H}\}$.
3. Consider $V/\sim$ where we write $p \sim q$ if there exists a $\phi : (SZ, d+p) \to (SZ, d+q)$ as in $(*)$.
4. Consider $Aut\,\mathcal{H}\backslash V/\sim$ as a "moduli space" to be called $M_{\mathcal{H}}$.

Here $Aut\,\mathcal{H}$ acts as follows: If $\rho : (SZ, d) \to \mathcal{H}$ is a Tate-Jozefiak resolution of $\mathcal{H}$ and $g \in Aut\,\mathcal{H}$, then $g\rho$ is also a resolution and hence g lifts to an automorphism $\bar{g} : (SZ, d) \to (SZ, d)$. Now if $(SZ, d+p)$ is a perturbation, so is $(SZ, d + \bar{g}p\bar{g}^{-1})$ and $p \to \bar{g}p\bar{g}^{-1}$ is the action of $Aut\,\mathcal{H}$ on V. The topology on $Aut\,\mathcal{H}\backslash V/\sim$ will turn out to have an invariant meaning, so that this quotient can meaningfully be called **the space of homotopy types with cohomology algebra $\mathcal{H}$**.

If $\mathcal{H}$ is finite dimensional, we will see fairly easily that V is an algebraic variety and the equivalence is via a group action of a unipotent algebraic group.

Consider the dg Lie algebra $Der\,SZ = \bigoplus_i Der^i SZ$ where $Der^i\,SZ$ is the sub–dg Lie algebra consisting of all derivations that raise total degree by the integer i. Define $L \subset Der\,SZ$ to consist of all derivations which decrease weight = total degree minus resolution degree. Any $\phi \in L$ of a particular total degree can be regarded as an infinite sum (and conversely)

$$\phi = \phi_1 + \phi_2 + \cdots + \phi_k + \ldots$$

where ϕ_k decreases weight by k. The infinite sum causes no problem because there are no elements of negative weight and, for any $z \in SZ$ of fixed weight $k \geq 0$, $\phi(z)$ will be a finite sum: $\phi_1(z) + \ldots \phi_{k+1}(z)$. In other words, L is complete with respect to the weight filtration.

The weights on L above allow us to analyze further $V \subset L^1$, i.e.,

$$V = \text{ the variety } \{p \mid (d+p)^2 = 0\}.$$

Theorem 2.3. *If $\mathcal{H}$ is finite dimesional, V is the cone on a projective variety.*

Proof. The point is that, if Z is finite dimensional, so will be L^1. Since d preserves weight, the equation $(d+p)^2 = 0$ is weighted homogeneous. That is, writing $p = \Sigma\, p_i$ where p_i decreases weight by i, we have that $\bar{t}p = \Sigma\, t^i p_i$ satisfies $(d + \bar{t}p)^2 = 0$, as can be seen by expanding and collecting terms of equal weight. The locus of this ideal

$$I = \{d + p \mid (d+p)^2 = 0\}$$

of weighted homogeneous polynomials is a subvariety of a (finite dimensional) projective subspace. $\square$

If $\mathcal{H}$ is only of finite type (i.e. finite dimensional in each degree), then V will be a pro-algebraic group acted upon by a pro-unipotent group.

As (SZ, d) is a model for (the cochains of) a simply connected space X of finite type, we can interpret $H(Z, d)$ as dual to $\pi_*(X)$ where d acts on Z as the *indecomposables*, i.e. the quotient $S^+Z/S^+Z \cdot S^+Z$ ([63] p. 301). For the restriction for Z to be finite dimensional, it is sometimes more appropriate for us to model the space X via a differential graded Lie algebra. We next consider several aspects of the theory of dg Lie algebras and return to the classification in §4.

3 Differential graded Lie algebras, models and perturbations

Differential graded Lie algebras appear in our theory in two ways, as models for spaces and as the graded derivations of either a dgca or of another dg Lie algebra (or the coalgebra analogs). We will be particularly concerned with certain standard constructions C and L which provide adjoint functors from the homotopy category of dg Lie algebras to the homotopy category of dgcc's and vice versa [46, 51].

CONNECTIVITY ASSUMPTIONS

Quillen's approach to rational homotopy theory is to construct a functor from simply connected rational spaces to dg Lie algebras and then apply C to obtain a dgcc model. The functor $A(\quad) = Hom(C(\quad), \mathbf{Q})$ from dg Lie algebras to dga's fits more readily into a traditional exposition, but the usual subtleties of the Hom functor necessitate the detour into the more natural differential graded coalgebras. We recall definitions and many of the basic results from Quillen [51], especially Appendix B.

We will also be concerned with perturbations of filtered dg Lie algebras. We conclude this section with a comparison of Der L and Der $A(L)$. (Recall all vector spaces are over $\mathbf{Q}$.)

A crucial motivation for Quillen's theory is Serre's result [59] that $\pi_*(\Omega X) \otimes \mathbf{Q}$ is isomorphic as a graded Lie algebra to the primitive subspace $P\, H_*(\Omega X, \mathbf{Q})$, so Quillen uses dg Lie algebras with lower indices and d of degree -1. We do not follow this tradition, but rather, consistent with our cohomological point of view, our dg Lie algebras will have upper indices and differentials d of degree $+1$. [1] (The history of graded Lie algebras is intimately related to homotopy theory, see [21].)

Definition 3.1. *[51], p. 209. A* **differential graded Lie algebra** *(dgL) L consists of*

1. *a graded vector space $L = \{L^i\},\ i \in [Z]$,*
2. *a graded Lie bracket $[\ ,\ ] : L^i \otimes L^j \to L^{i+j}$ such that*

$$[\theta, \phi] = -(-1)^{ij}[\phi, \theta] \qquad \text{and}$$
$$(-1)^{ik}[\theta,[\phi,\psi]] + (-1)^{ji}[\phi,[\psi,\theta]] + (-1)^{kj}[\psi,[\theta,\psi]] = 0,$$

   (In other words, $ad\,\theta := [\theta,\ ]$ is a derivation of degree i for $\theta \in L^i$.)

3. *a graded Lie derivation $d : L^i \to L^{i+1}$ such that $d^2 = 0$.*

Definition and Example 3.2. *For any dga $(A = \oplus A^i, d_A)$, we have the dg Lie algebra $L = Der\, A = \oplus Der^i\, A$ where $Der^i\, A = \{$derivations $\theta : A \to A$ of degree $i\}$. The bracket is the graded commutator: $[\theta, \phi] = \theta\phi - (-1)^{ij}\phi\theta$ for $\theta \in Der^i\, A$ and $\phi \in Der^j\, A$. We have the differential $d_A \in Der^1\, A$ and $d_L(\theta) := [d_A, \theta] := d_A\theta - (-1)^i\theta d_A$, i.e., $d_L = ad(d_A)$.*

[1] The two traditions are identified via the convention: $L^i = L_{-i}$.

Note that i ranges over all integers, not necessarily just positive or just negative; later we will have to consider bounds. Even if A is of finite type, Der A need not be.

Of course, for any dg Lie algebra L, the homology $H(L)$ is again a graded Lie algebra, but there is more structure than that inherited for L (see 6).

Example 3.3. *For the special case of the Tate-Jozefiak resolution* (SZ, d) *of* $\mathcal{H}$*, we are interested in the sub–Lie algebra* $L \subset \mathit{Der}\, SZ$ *consisting of all derivations which decrease weight.*

Definition and Example 3.4. *For any dg Lie algebra* $(L = \oplus L^i, d_L)$*, we have the dg Lie algebra* $\mathit{Der}\, L = \oplus \mathit{Der}^i\, L$ *where* $\mathit{Der}^i\, L = \{\text{derivations } \theta : L^j \to L^{j+i}\}$ *with again the graded commutator bracket and the induced differential* $d(\theta) := [d_L, \theta] := d_L\theta - (-1)^i \theta d_L$*.*

3.1 Differential graded commutative coalgebras and dg Lie algebras

In order to dualize conveniently, and motivated by the case of the (admittedly not commutative) chains of a topological space, we cast our definition of dgcc in the following form.

Definition 3.5. *A* **dgcc d***ifferential* **g***raded (***c***ommutative* **c***oalgebra) consists of*

1. *a graded vector space* $C = \{C^n,\, n \in Z\}$*,*
2. *a differential* $d : C^n \to C^{n+1}$ *and*
3. *a differential map* $\Delta : C \to C \otimes C$ *called a* **comultiplication** *which is associative and graded commutative and*
4. *a counit* $\epsilon : C \to \mathbf{Q}$ *such that* $(\epsilon \otimes 1)\Delta = (1 \otimes \epsilon)\Delta = id_C$*.*

We say C *is* **augmented** *if there is given a dgc map* $\eta : \mathbf{Q} \to C$ *and that* C *is* **connected** *if* $C^n = 0$ *for* $n < 0$ *and* $C = \mathbf{Q}$*.*

The dual $Hom\,(C, \mathbf{Q})$ of a connected dgcc is a connected dgca with $A^{-n} = Hom\,(C^n, \mathbf{Q})$. Conversely, if A is a dgca of finite type (meaning that each A^{-n} is finite dimensional over $\mathbf{Q}$), then $Hom\,(A, \mathbf{Q})$ inherits the structure of a dgcc.

Most of our constructions make use of the tensor algebra and tensor coalgebra; we pause to review structure, notation and nomenclature.

Let M be a graded $\mathbf{Q}$-vector space; it generates free objects as follows:

3.2 The tensor algebra and free Lie algebra

.

The tensor algebra $T(M) = \bigoplus_{n\geq 0} M^{\otimes n}$ where $M^{\otimes 0} = \mathbf{Q}$ and $M^{\otimes n} := M \otimes \cdots \otimes M$ with $(a_1 \otimes \cdots \otimes a_p)(a_{p+1} \otimes \cdots \otimes a_{p+q}) = a_1 \otimes \cdots \otimes a_{p+q}$ is the free associative algebra generated

by M. It is the free graded associative algebra generated by M with respect to the **total** grading, which is $\Sigma(|a_i|)$ where $|a_i|$ is the grading of a_i in M.

The free graded Lie algebra $L(M)$ can be realized as a Lie sub-algebra of $T(M)$ as follows: Regard $T(M)$ itself as a graded Lie algebra under the *graded commutator*

$$[x,y] = x \otimes y - (-1)^{deg\, x deg\, y} y \otimes x,$$

then the Lie sub-algebra generated by M is (isomorphic to) the free Lie algebra $L(M)$. In characteristic 0, this can usefully be further analyzed (cf. Friedrichs' Theorem [50, 58]) by considering $T(M)$ as a Hopf algebra with respect to the *unshuffle* diagonal

$$\triangle(a_1 \otimes \cdots \otimes a_n) = \sum_{(p,q)-\text{shuffles}\,\sigma} (-1)^\sigma (a_{\sigma(1)} \otimes \cdots \otimes (a_{\sigma(p)}) \otimes (a_{\sigma(p+1)} \otimes \cdots \otimes a_{\sigma(n)})$$

where σ being an unshuffle (sometimes called a shuffle!) means $\sigma(1) < \sigma(2) < \cdots < \sigma(p)$ and $\sigma(p+1) < \cdots < \sigma(n)$ and $(-1)^\sigma$ is the sign of the graded permutation. The free graded Lie algebra $L(M)$ is then isomorphic to the algebra of primitives, $P(T(M))$, i.e., $x \in T(M)$ is **primitive** if and only if $\triangle x = x \otimes 1 + 1 \otimes x$. The Hopf algebra $T(M)$ is isomorphic to the universal enveloping algebra $U(L(M))$ [51].

Convention. The **shift** functor s on differential graded objects shifts degrees down by one for algebras and up by one for coalgebras:

$$s : A^{q+1} \simeq (sA)^q and$$
$$s : C^{q-1} \simeq (sC)^q,$$

while the differential anti-commutes with s, i.e. $ds = -sd$.

Definition and Example 3.6. *[51] p. 290. For any augmented differential graded coalgebra (dgc) C with commutative diagonal Δ, let $L(C)$ be the differential graded Lie algebra which is the free graded Lie algebra on $s\bar{C} = s(C/\mathbf{Q})$ with differential d which is the derivation determined by d_C and Δ, i.e.,*

$$d(sc) = -s(d_C c) + \frac{1}{2}\Sigma(-1)^{deg\, c_i}[sc_i, sc_i']$$

where $\Delta c = \Sigma c_i \otimes c_i'$ or, in Heyneman-Sweedler notation, $\Delta c = \Sigma c_{(1)} \otimes c_{(2)}$, omitting the terms $1 \otimes c$ and $c \otimes 1$.

An enlightening alternative description is that $L(C)$ can be identified with $P\Omega C$, the space of primitive elements in the cobar construction on the commutative coalgebra C [1]. That is, ΩC is the graded Hopf algebra $T(s\bar{C})$ described above with differential which is the derivation determined by

$$d(sc) = -s(d_C c) + \Sigma(-1)^{deg\, c_i} sc_i \otimes sc_i'.$$

In particular, we wish to apply this construction L to the dual $\mathcal{H}_*$ of a graded commutative algebra $\mathcal{H}$. Thinking of $\mathcal{H}$ as a cohomology algebra and assuming $\mathcal{H}$ connected of finite type, the graded linear dual $\mathcal{H}_*$ we then regard as homology,

The coalgebra grading is given by $(\mathcal{H}_*)^{-n} = Hom\,(\mathcal{H}^n, \mathbf{Q})$, which can also be written as $\mathcal{H}_n$.

By abuse of notation, we will write $L(\mathcal{H})$ instead of $L(\mathcal{H}_*)$. The differential above then becomes

$$d(s\phi)(su \otimes sv) = \pm\phi(uv)$$

for $u, v \in \mathcal{H}$ and $\phi : \mathcal{H}^n \to \mathbf{Q}$.

Henceforth, we rely heavily on our assumption that all (non–differential) cga's $\mathcal{H}$ are simply connected of finite type. Thus $L(\mathcal{H})$ will be of finite type.

Models for dgc algebras can be obtained conveniently via the functor C adjoint to L. For ordinary Lie algebras L, the construction $C(L)$ reduces to the complex used by Cartan–Chevalley–Eilenberg [7] and Koszul [32] to define the homology of Lie algebras.

3.3 The tensor coalgebra and free Lie coalgebra

Again let M be a graded $\mathbf{Q}$-vector space. Consider the tensor coalgebra $T^c(M) = \underset{n\geq 0}{\oplus} M^{\otimes n}$ with the *deconcatenation* or *cup coproduct*

$$\Delta(a_1 \otimes \cdots \otimes a_n) = \underset{p+q=n}{\Sigma} (a_1 \otimes \cdots \otimes a_p) \otimes (a_{p+1} \otimes \cdots \otimes a_n).$$

Remark 3.7. *Some authors refer to $T^c(M)$ as the cofree associative unitary coalgebra cogenerated by M, but it is cofree only as a pointed irreducible coalgebra. On the other hand, others refer to the 'tensor coalgebra' meaning the completion $\hat{T}^c(M)$ which is the cofree associative unitary coalgebra, see below and [43] Remark 3.52.*

As for the tensor algebra, the total grading $\Sigma(|a_i| + 1)$ where $|a_i|$ is the grading of a_i in M is usually most important, but the number of tensor factors provides a useful decreasing filtration: $\mathcal{F}_p T^c(M) = \underset{n\geq p}{\oplus} M^{\otimes n}$. For our purposes, it is important to pass to the completion $\hat{T}^c(M)$ of $T^c(M)$ with respect to that filtration: $\hat{T}^c(M) \cong \prod_{n\geq 0} M^{\otimes n}$. The diagonal Δ on $T^c(M)$ extends to $\Delta : \hat{T}^c(M) \to \hat{T}^c(M)\hat{\otimes}\hat{T}^c(M)$, the completed tensor product being given by the completion with respect to $F_n(\hat{\otimes}) = \underset{p+q\geq n}{\oplus} F_p \otimes F_q$. With this structure, $\hat{T}^c(M)$ is universal with respect to the category of cocomplete connected graded coalgebras (cf.[43] for a very thorough treatment or [58] for the algebra version and [51] Appendix A for connected coalgebras).

Definition 3.8. *A* **cocomplete** *dgc (C, d, Δ) consists of*

1. *a decreasingly filtered dg vector space C which is cocomplete (i.e., $C = \varprojlim C/F_pC$), and*

2. *a filtered (= continuous) chain map $\Delta : C \to C\hat{\otimes}C$ which is associative.*

Morphisms of cocomplete dgcc's respect the given filtrations.

Thus $(\hat{T}^c(M), \Delta)$ is the cofree graded cocomplete associative coalgebra cogenerated by M (cf. [42]). That is, $(\hat{T}^c(M), \Delta)$ has the following universal property:

$$Coalg(C, \hat{T}^c(M) \cong Hom(C, M)$$

for cocomplete associative coalgebras C, where $Coalg$ denotes morphisms of such coalgebras while Hom denotes linear maps of $\mathbf{Q}$-vector spaces. In other words, the functor $(\hat{T}^c(M), \Delta)$ is adjoint to the forgetful functor from graded cocomplete associative coalgebras to graded vector spaces.

A major attribute of cocomplete coalgebras and $\hat{T}^c(M)$ in particular is that their space of group-like elements, $\mathcal{G}C := \{c|\Delta c = c\hat{\otimes}c\}$ need not be spanned by $1 \in \mathbf{Q}$. There is the obvious $1 \in \mathbf{Q} = M^{\otimes 0}$ with $\Delta 1 = 1 \otimes 1$ but for any $p \in M$, the element $q = 1 + p + p \otimes p + p \otimes p \otimes p + \dots$ also has $\Delta q = q \otimes q$.

The cofree graded Lie coalgebra can be realized as a quotient of $T^c(M)^+$, where $+$ denotes the part of strictly positive $\otimes$–degree, i.e. F_1. The tensor coalgebra $T^c(M)$ can be given a Hopf algebra structure by using the shuffle multiplication, i.e.,

$$(a_1 \cdots \otimes a_p) * (b_1 \otimes \cdots \otimes b_q) = \sum_\sigma (-1)^\sigma c_{\sigma(1)} \otimes \cdots \otimes c_{\sigma(p+q)}$$

where σ is a (p,q)-shuffle permutation as above and $c_1 \otimes \cdots \otimes c_{p+q}$ is just $a_1 \otimes \cdots \otimes a_p \otimes b_1 \otimes \cdots \otimes b_q$. The Lie coalgebra $L^c(M)$ consists of the indecomposables of this Hopf algebra, i.e., $T^c(M)^+/T^c(M)^+ * T^c(M)^+$. The Hopf algebra $T^c(M)$ is, provided M is the universal enveloping coalgebra of $L^c(M)$; the associated graded of $T^c(M)$ is, as algebra, isomorphic to $SM)$, the free graded commutative algebra on M.

This discussion can be 'completed' by using $\hat{T}^c(M)^+$ and the desired universal property follows:

$$Liecoalg(C, \hat{L}^c(M) \cong Hom(C, M)$$

for cocomplete Lie coalgebras C. The cofree graded cocomplete commutative coalgebra $S^c(M)$ generated by M is the maximal commutative sub–coalgebra of $T^c(M)$, i.e., the cocomplete sub–coalgebra of all graded symmetric tensors (invariant under signed permutations). The cofreeness of $S^c(M)$ means that a coalgebra map f from a connected commutative coalgebra C into $S^c(M)$ is determined by the projection $\pi f : C \to M$ and the same is true with respect to coderivations $\theta : S^c(M) \to S^c(M)$: θ is determined by the projection $\pi\theta : S^c(M) \to M$.

Recall that a **coderivation** θ of a coalgebra C is a linear map $\theta : C \longrightarrow C$ such that $(\theta \otimes 1 + 1 \otimes \theta)\Delta = \Delta\theta$.

3.4 The standard construction $C(L)$

The Cartan–Chevalley–Eilenberg chain complex [7, 32] generalizes easily to graded Lie algebras and even dg Lie algebras .

Definition 3.9. *([51] p. 291). Given a dg Lie algebra (L, d_L), let $C(L)$ denote the free cocomplete commutative coalgebra $S^c(sL)$ with total differential cogenerated as a graded coderivation by d_L and $[\ ,\ ]$, meaning that it is the graded coderivation such that:*

$$d(s\theta) = -s(d_L\theta)$$

and

$$d(s\theta \wedge s\phi) = s[\theta, \phi] - s(d\theta) \wedge s\phi - (-1)^{|s\theta|} s\theta \wedge s(d\phi)$$

(here $s\theta \wedge s\phi$ is the symmetric tensor $s\theta \otimes s\phi + (-1)^{|s\theta|\,|s\phi|} s\theta \otimes s\phi$).

If L is ungraded, $C(L)$ is precisely the Cartan–Chevalley–Eilenberg chain complex [7, 32] and $H(C(L))$ is the Lie algebra homology $H_*^{Lie}(L)$. We will exhibit this in more detail below for the cochain complex and cohomology.

Definition 3.10. *The* **homology** *$H^{dg\ell}(L)$ is the graded coalgebra $H(C(L), d_C)$ with the grading and comultiplication inherited from $C(L)$. The decoration $H^{dg\ell}$ indicates this is the homology of L qua differential Lie algebra, i.e., in the sense of category theory or homological algebra.*

Theorem 3.11. *([51] Appendix B) C and L are adjoint functors between the homotopy category of dg Lie algebras with $L_q = 0$ for $q \leq 0$ and the homotopy category of simply connected dgcc's. The adjunction morphisms $\alpha : LC(L) \to L$ and $\beta : C \to CL(C)$ induce isomorphisms in homology.*

Extending the terminology from dgca's (as in [4]), we speak of $\alpha : LC(L) \to L$ as a **model** for L and of $\beta : C \to CL(C)$ as a **model** for C. If $C = C(X)$ is a commutative chain coalgebra over $\mathbf{Q}$ of a simply connected topological space X, we speak of $C \to L(C)$ as a dg Lie algebra model for X.

We will often find it useful to replace L by a simpler dg Lie algebra K with the same homology and then will want to compare $C(L)$ and $C(K)$.

Theorem 3.12. *[51]. If $f : L \to K$ is a map of dg Lie algebras which are positively graded and $H(f) : H(L) \simeq H(K)$, then the induced map $H(C(f)) : H^{dg\ell}(L) \to H^{dg\ell}(K)$ is an isomorphism; i.e., $C(L) \to C(K)$ is a homology equivalence/quasi-isomorphism.*

To compare the construction for more general dg Lie algebras will be important for our homotopy classification.

3.5 The Quillen and Milnor/Moore et al spectral sequences [51, 44, 45]

The coalgebra $C(L)$ is equipped with a natural increasing filtration, the tensor degree, i.e., $s\theta_1 \otimes \cdots \otimes s\theta_n$ (where $\theta_i \in L$) has filtration $p \geq n$. The associated spectral sequence has

$$E = (S^c(sL), d_L) \quad \text{so that} \tag{1}$$

$$E_1 = (C(H(L)), d_1 = [\ ,\ ]) \tag{2}$$

and hence E_2 is the Lie algebra homology of $H(L)$, while the spectral sequence abuts to $H^{dg\ell}(L)$.

Compare the spectral sequence for relating the homology of a loop space ΩX to that of X as its classifying space. In that situation, E_2 is the associative algebra homology of $H(\Omega X)$, that is, $Tor\, H^{(\Omega X)}(\mathbf{Q},\mathbf{Q})$, the homology of $T^c(s\bar{H}(\Omega X))$, with d_1 determined by the loop multiplication m_*. But over the rationals, Quillen has constructed a dg Lie algebra λ_X such that $H(\lambda_X) \simeq \pi_*(\Omega X) \otimes \mathbf{Q}$ and $C(\lambda_X)$ is homotopy equivalent to the coalgebra of chains on X. Moreover, over the rationals, Serre [59] has shown that $H(\Omega X) \simeq U(\pi_*(\Omega X) \otimes \mathbf{Q})$, the universal enveloping algebra on the Lie algebra $\pi_*(\Omega X) \otimes \mathbf{Q}$. Thus comparing Quillen and Milnor–Moore at the E_2 level we have

$$H^{Lie}(\pi_*\Omega X \otimes \mathbf{Q}) \simeq H^{assoc}(U(\pi_*\Omega X \otimes \mathbf{Q}))$$

by a well–known result in homological algebra, while the spectral sequence abuts to

$$H(C(\lambda_X)) \simeq H(X).$$

To fix ideas and for later use, we consider a special case in which $H(L) = L(V)$, the free Lie algebra on a positively graded vector space V. Since $H(L)$ is free, we can choose representative cycles and hence a dg Lie algebra map $H(L) \to L$ which is a homology isomorphism. Thus we have isomorphic spectral sequences, but for $H(L)$ the spectral sequence collapses: $E_\infty \approx E_2 \approx H(S^c sH(L))$. That is, since d_1 is $[\ ,\ ]$ and $H(L)$ is free, only the $[\ ,\ ]$-indecomposables of $H(L)$ survive, i.e.,

$$H^{dg\ell}(L) \approx H(S^c sH(L)) \approx \mathbf{Q} \otimes sV$$

with sV primitive.

Use of this spectral sequence implies (compare Quillen, Appendix B [51]:

Theorem 3.13. *If $f : L \to K$ is a map of dg Lie algebras which are connected and $H(f) : H(L) \cong H(K)$, then $H(C(f))$ is an isomorphism.*

3.6 The standard construction $A(L)$

The dual of $C(L)$ is a dgca which we denote by $A(L) = Hom\,(C(L), \mathbf{Q})$.

We can interpret $A(L)$ in terms of the (set of) alternating forms on L: for any L–module M, a linear homomorphism $C(L) \to M$ of degree q can be regarded as an alternating multilinear form on L. (Only if $C(L)$ is of finite dimension in each degree, e.g. if L is non-negatively graded of finite type, should we think of $A(L)$ as $S(sL^{\#})^2$. [2] The coboundary on $A(L)$ can then be written explicitly as

$$\begin{aligned}(d_A f)(X_1,\dots,X_n) = \Sigma \pm f(\dots, dX_i, \dots)&\\ \pm \Sigma f([\hat{X}_i, \hat{X}_j], \dots, X_i, \dots, X_j, \dots)&\\ \pm \Sigma X_i \circ f(X_1, \dots, X_i, \dots).&\end{aligned}$$

[2] With the exception of (co)algebras that are interpreted as, vice versa, (co)homology, we will usually denote duals by $^{\#}$.

For future reference, we write $d_A = d' + d''$ corresponding to the first term and the remaining terms above.

If L is an ordinary (ungraded) Lie algebra, $d' = 0$ and d'' is (up to sign) the differential used by Chevalley–Eilenberg and Koszul.

Definition 3.14. *For a dg Lie algebra L, the* **cohomology** $H^*_{dg\ell}(L)$ *is the algebra* $H(A(L)) = H(Hom\,(C(L), \mathbf{Q})$.

The adjointness of L and C will show that, given suitable finiteness conditions, $A(L(\mathcal{H}))$ is a model for $\mathcal{H}$, i.e., a (possibly non–minimal) (SZ, d) resolution of $\mathcal{H}$ as in Chapter 2. Because of the shift in grading and the way degrees add, L must be of finite type and suitably bounded for $C(L)$ to be of finite type: $L^n = 0$ for $n \leq 0$ or $n \geq 2$. For example, if $\mathcal{H}$ is simply connected and of finite type, then $C(L(\mathcal{H}))$ is of finite type.

3.7 Comparison of Der L and Der $C(L)$

We are interested in comparing perturbations of $A(L)$ with the corresponding changes in $L(\mathcal{H})$.

Definition 3.15. *A* **perturbation** *of $L(\mathcal{H})$ is a Lie derivation p of the same degree as d such that p increases bracket length by at least 2 and $(d + p)^2 = 0$.*

We are interested in derivations of $A(\pi)$ where π is a dg Lie algebra. We will be using L to denote $Der\,\pi$. Although somewhat unfamiliar, the dg Lie algebra of coderivations of $C(\pi)$ turns out to be more susceptible of straightforward analysis.

The graded space of all graded coderivations of C will be denoted $Der\,C$. Later we will examine $A(\pi)$ directly, under suitable finiteness conditions.

Definition 3.16. *For a dg Lie algebra π, the* **semidirect product** $s\pi \sharp Der\,\pi$ *is, as* $\mathbf{Q}$-*vector space,* $s\pi \oplus Der\,\pi$. *As a graded Lie algebra, it has $s\pi$ as an abelian sub–algebra and $Der\,\pi$ as a subalgebra which acts on $s\pi$ by derivations via $[\phi, s\theta] = (-1)^{\phi} s\phi(\theta)$. The differential $d_\sharp$ is given by $d_\sharp(s\theta) = -sd_\pi\theta \oplus ad\,\theta$ for $\theta \in \pi$, $\phi \in Der\,\pi$.*

Theorem 3.17. *For any dg Lie algebra π with $\pi_i = 0$ for $i \leq 0$, there is a canonical map*

$$\rho : s\pi \sharp Der\,\pi \to Der\,C(\pi)$$

of dg Lie algebras. If π is free as a graded Lie algebra, then ρ is a homology isomorphism. (If π is free on more than one generator, then $s\pi \sharp Der\,\pi \to Der\,\pi/ad\,\pi$ is also a homology isomorphism.)

Proof. Since $C(\pi)$ is cofree on $s\pi$, a coderivation of $C(\pi)$ is determined by its projection onto $s\pi$. Thus $Der\,C(\pi)$ is isomorphic to $Hom\,(C(\pi), s\pi)$. Moreover, this is an isomorphism of dg $\mathbf{Q}$–modules precisely if $Hom\,(C(\pi), s\pi)$ is given the Chevalley–Eilenberg differential (3.11), where π is regarded as a π–module by the adjoint action, i.e., $ad\,x : y \to [x, y]$. We

can define ρ via this identification. The coderivation $\rho(sx)$ is determined by projecting $C(\pi)$ onto $\mathbf{Q}$ (by the counit) and then mapping to sx, while for $\theta \in Der\,\pi$, correspondingly $\rho(\theta)$ is determined by projecting $C(\pi)$ onto $s\pi$ and then composing with $s\theta : s\pi \to s\pi$. A careful check shows ρ is a map of dg Lie algebras .

To calculate $H(\rho)$, notice that $\rho|Der\,\pi \subset Hom\,(s\pi, s\pi)$ and, in fact, lies in the kernel of part of the differential. That is, for $sh \in Hom\,(s\pi, s\pi)$,

$$d''sh(s[x_1, x_2]) = s(h[x_1, x_2] - [x_1, h(x_2)] + (-1)^{x_1 x_2}[x_2, h(x_1)])$$

which is zero if and only if h is a (graded) derivation of π. Thus, with regard to tensor degree, $H(\rho)$ is an isomorphism in tensor degrees 0 and 1. If $d_\pi = 0$, then π being free implies $H^{Lie}(\pi;\pi) = 0$ above tensor degree 1 [24].

For a general dg Lie algebra π, we use a spectral sequence comparison. Filter $Der\,C(\pi)$ by internal degree, i.e., $\theta \in Der\,C(\pi) \cong Hom\,(C(\pi), s\pi)$ is of filtration $\leq q$ if proj $\circ\, \theta(sx_1 \wedge \cdots \wedge sx_n)$ is of deg $\leq \Sigma \deg x_i + q$. Thus the associated graded coalgebra has d_π equivalent to zero and ρ induces an isomorphism of E_1 terms for π free. Since E_1 is concentrated in complementary (= tensor) degrees 0 and 1, the homology isomorphism follows.

Finally, if π is free on more than one generator, then the center of π is 0, so that the sub–dg Lie algebra $s\pi \sharp\, ad\,\pi$ of $s\pi \sharp\, Der\,\pi$ has 0 homology. The exact sequence

$$0 \to s\pi \sharp\, \mathrm{ad}\,\pi \to s\pi \sharp\, Der\,\pi \to Der\,\pi/\mathrm{ad}\pi \to 0$$

now yields $H(s\pi \sharp\, Der\,\pi) \cong H(Der\,\pi/\mathrm{ad}\,\pi)$. □

3.8 $A(L(\mathcal{H}))$ and filtered models

We will be interested in $A(L(\mathcal{H}))$ **only** if $\mathcal{H}$ is simply connected ($\mathcal{H} = \bigoplus_{i>1} \mathcal{H}^i$) and of finite type. The dgc $C(L(\mathcal{H}))$ is then of finite type and we can then describe $A(L(\mathcal{H}))$ usefully without dualizing twice. Earlier, we described the cofree Lie coalgebra as the indecomposable quotient of the tensor coalgebra. The construction $A(L(\mathcal{H}))$ for **simply connected $\mathcal{H}$ of finite type** can be described as the free commutative algebra on (s of) the free Lie coalgebra on $\mathcal{H}$. A typical element of $A(L(\mathcal{H}))$ is then a sum of symmetric tensors $a_1 \wedge \cdots \wedge a_n$ where each a_i is a sum of terms

$$\Sigma(-1)^\sigma [sh_{\sigma(1)}| \dots |sh_{\sigma(k)}].$$

The differential d is a derivation generated by the bracket in $L(\mathcal{H})$ and

$$m : [h_1| \dots |h_k] \to \Sigma(-1)^i [h_1| \dots |h_i h_{i+1}| \dots |h_k].$$

The obvious algebra map $A(L(\mathcal{H})) \to \mathcal{H}$ determined by $[h] \to h$ and

$$[h_1| \dots |h_k] \to 0, \qquad \text{for } k > 1$$

is a resolution with $k - 1$ as the **resolution** degree. The total degree of $[h_1| \dots |h_n]$ is $\Sigma\, deg\, h_i - k + 1$; thus the **weight** is $\Sigma\, deg\, h_i$.

The resolution $A(L(\mathcal{H})) \to \mathcal{H}$ is the Tate–Jozefiak resolution if and only if $\mathcal{H}$ has trivial products. In general, the Tate–Jozefiak resolution is a minimal model for $A(L(\mathcal{H}))$, but more can be said because $A(L(\mathcal{H})) \to \mathcal{H}$ is a filtered model if we use the filtration by weight.

Definition 3.18. *A* **filtered model** $(SZ, d) \to A$ *of a dgca* A *is a model with a dga filtration such that* $E_1(SZ, d) \cong H(A)$ *is concentrated in filtration* 0.

The comparison theorem of Halperin and Stasheff [20] generalizes directly to filtered models, given an equivalence $(SZ, d) \to A(L(\mathcal{H}))$ with the Tate–Jozefiak model which respects filtration. In §6, we will consider perturbations of $A(L(\mathcal{H}))$ as an alternative method of classifying homotopy types. By using $A(L(\mathcal{H}))$, the problem can be further reduced to perturbations in $Der\, L(\mathcal{H})$. In particular, a perturbation which decreases weight by i will be represented in terms of maps of a subspace of $\mathcal{H}^{\otimes i+2}$ into $\mathcal{H}$ of degree $-i$, which is suggestive of a Massey product (as further explained in §8).

As one would hope, the classification does not depend on the model used. As a first step toward this independence, we compare Lie algebras of derivations.

Theorem 3.19. *Let* $M \to A$ *be the minimal model for a simply connected dgca* A*, free as a gca.. There is an induced map* $Der\, M \to Der\, A$ *of dg Lie algebras which is a homology isomorphism (quasi-isomorphism).*

Proof. According to Sullivan [63] p. ???, A splits as $M \otimes C$ as dgcas with C contractible. The algebras M, C and A are free on differential graded vector spaces X, Y and $X \oplus Y$ respectively with Y contractible. We have a sequence of maps $DerB =$

$$Hom(X, M) \to Hom(X, M \otimes C = A) \to Hom(X, A) \oplus Hom(Y, A) = Hom(X \oplus Y, A) = DerA.$$

Since X, Y and $X \oplus Y$ are graded vector spaces, we can use the identity $H(Hom(U, V)) \simeq Hom(HU, HV)$ to conclude that each of the above maps is a quasi-isomorphism. $\square$

Thus the dg Lie algebra $Der\, A$ is a “ homology invariant” of the free simply connected dga A; its cohomology does not depend on the choice of A. We may always choose a model $L = s\pi \sharp\, Der\, \pi$ for $Der\, A$ $\quad(\pi = \tilde{L}H(A)$ with suitable differential), which will have finite type if $dim\, H(A) < \infty$.

Notice further that if A is a filtered model, then the map $(SZ, d) \to A$ in the theorem preserves the filtration; in particular, this is true for the weight filtration.

Let $W_A \subset Der\, A$ denote the sub–dg Lie algebra of weight decreasing derivations. Now compare $W_{(SZ,d)} \to W_A$ as above. If $\theta \in Der\,(B \otimes C)$ is weight decreasing, then so is ϕ, since d preserves weight. Thus we have:

Theorem 3.20. *If* $SZ \to A$ *is the minimal model for a filtered free dgca* A*, then the induced map* $W_{SZ} \to W_A$ *is a homology isomorphism.*

Our classification of homotopy types will proceed via the classification of perturbations in such a way that it will depend on only the homotopy type of W_A as a dg Lie algebra

and thus can be analyzed in terms of the minimal model or $A(L(\mathcal{H}))$. The advantage of the latter is that we can further reduce the problem to perturbations of $L(\mathcal{H})$ with respect to the bracket length as weight.

Recall $L(\mathcal{H})$ is a free dg Lie algebra naturally filtered by bracket length. The complementary degree is the sum of the degrees in $\mathcal{H}$, which generates the weights in $A(L(\mathcal{H}))$; we refer to this sum as the weight in $L(\mathcal{H})$ also.

Thus a perturbation of $L(\mathcal{H})$ generates one in $A(L(\mathcal{H}))$ and indeed we have a natural map $W_{L(\mathcal{H})} \to W_{A(L(\mathcal{H}))}$ of dg Lie algebras .

Theorem 3.21. *For simply connected $\mathcal{H}$ of finite type, the natural map $W_{L(\mathcal{H})} \to W_{A(L(\mathcal{H}))}$ is a homology isomorphism.*

Proof. Under the given hypotheses on $\mathcal{H}$, the analog of Theorem 3.17 implies there is a homology isomorphism

$$sL(\mathcal{H}) \sharp\, Der\, L(\mathcal{H}) \to\ Der\, A(L(\mathcal{H})).$$

The factor $sL(\mathcal{H})$ maps to derivations of $A(L(\mathcal{H}))$ as follows: For $x \in L(\mathcal{H})$, the derivation θ_{sx} of $A(L(\mathcal{H}))$ is defined as the partial derivative with respect to sx. Thus θ_{sx} decreases the bracket length and hence increases weight. Thus the weight decreasing elements on the left hand side are precisely $W_{L(\mathcal{H})}$ and the proof of the analog of Theorem 3.17 restricts to the sub–dg Lie algebras W of weight decreasing derivations, since the differentials in $L(\mathcal{H})$ and $A(L(\mathcal{H}))$ preserve weight. □

Thus we turn to the classification of homotopy types having a variety of models to use.

4 Classifying maps of perturbations and homotopies: The Main Homotopy Theorem.

Motivated by the classification of fibrations (see §9) [60, 65, 2], we find we can classify perturbations (and therefore homotopy types) by the "path components" of a universal example. Although we originally tried to use a universal dgca, we gradually came to the firm conviction that **cocomplete** dgccs are the real classifying objects; the dual algebras work under suitable finiteness restrictions.

Main Homotopy Theorem 4.1. *Let $\mathcal{H}$ be a simply connected cga of finite type and $(SZ, d) \to \mathcal{H}$ a filtered model. The set of augmented homotopy types of dgca's $(A, i : \mathcal{H} \cong H(A)$ is in 1–1 correspondence with the path components of $C(L)$ where $L \subset Der\, SZ$ consists of the weight decreasing derivations and $\hat{C}(L)$ is as in* **??**.

The rationals $\mathbf{Q}$ as a coalgebra serve as a "point" and a "path" is to be a special kind of homotopy of $\mathbf{Q} \to \hat{C}(L)$, but homotopy of coalgebra maps is a subtle concept. For motivation, we first review homotopy of dgca maps.

Definition 4.2. *For dgca's A and B with A free, two dga maps $f, f_1 : A \to B$ are* **homotopic** *if there is a dga map $A \to B[t, dt]$ such that f_i is obtained by setting $t = i$, $dt = 0$.*

(For simply connected A, there is a completely equivalent definition [63, 6] in terms of dga maps $A^I \to B$ where A^I models the topological space of paths X^I.)

Remark. [6], p. 88: Such a homotopy does *not* imply there is a chain homotopy $h : A \to B$ such that $h(xy) = h(x)\, f_1(y) \pm f(x)\, h(y)$ which would be the dga version of "homotopic through multiplicative maps " but does imply that the induced maps of bar constructions

$$\mathbb{B}f_i : \mathbb{B}A \to \mathbb{B}B$$

are homotopic through coalgebra maps.

This is the notion of homotopy appropriate to specifying the uniqueness of perturbations.

Theorem 4.3. *Compare [20] p. 253-4: If $\pi_i : (SZ, d + p_i) \to (A, d_A)$ for $i = 0, 1$ are maps which induce ρ^* in bottom degree 0 and the p_i decrease weight, then there is an isomorphism and dga map $\phi : (SZ, d + p) \to (SZ, d + p_1)$ such that $\phi - Id$ decreases weight and $\pi_1\phi$ is homotopic to π.*

4.1 Homotopy of coalgebra maps

The algebra $I = S[t, dt]$ with t of degree 0 and dt of degree 1 is implicit in the above definition (4.2) of homotopy of dg algebra maps. We regard I as the dual of the coalgebra $I^{\#}$ with (additive) basis $\{t_i, t_i u | i = 0, 1, \dots\}$, diagonal $\Delta t_n = \sum\limits_{i+j=n} t_i \otimes t_j$ and coderivation differential $d(t_n u) = (n + 1)t_{n+1}$. We denote t by t_1 when convenient.

There are difficulties with defining homotopies of coalgebras as maps $C \otimes I_* \to D$ because the end of the homotopy corresponds to the image of $\Sigma\, t_i$.

Definition 4.4. *The completion of I_* with respect to the obvious filtration we will denote by J and refer to that completion as the* **unit interval coalgebra***. (We could call t_i the i–th copower of $t = t_1$, and J the coalgebra of formal copower series.) The symbol $\Sigma\, t_i$ does represent an element of J.*

Definition 4.5. *Given two cocomplete dgcc's C and D, two filtration preserving maps $f_i : C \to D$ are* **homotopic** *if there is a filtration preserving dgc map $h : C \hat{\otimes} J \to D$ such that $f(c) = h(c \otimes 1)$ and $f_1(c) = \Sigma\, h(c \otimes t_i)$.*

In particular, the two "endpoints" $\mathbf{Q} \to J$ given by $1 \to 1$ and $1 \to \Sigma\, t_i$ are homotopic.

A case of particular importance is that in which D is $\hat{C}(L)$, the cocompletion of $C(L)$, which can be regarded a a sub-coalgebra of $\hat{T}^c sL$.

Definition 4.6. *Let (L, d_L) be a dg Lie algebra and $p \in L_1$ such that $d_L p + \frac{1}{2}[p,p] = 0$. The* **classifying map**

$$\chi(p) : \mathbf{Q} \to C(L)$$

is the dgc map determined by

$$1 \to sp \in sL,$$

$$i.e., \chi(p)(1) = 1 + \Sigma\; sp^{\otimes n} = 1 + sp + sp \otimes sp + \cdots .$$

As $\hat{C}(L)$ is a sub-coalgebra of $\hat{T}^c sL$, so $\chi(p)$ is essentially $\exp(p)$. That $\chi(p)$ is a differential map is true precisely because $d_L p + \frac{1}{2}[p,p] = 0$, which is the same as $d_L p + p^2 = 0$.

We are concerned with the special case of dg homotopy between two classifying maps $\chi(p_i) : \mathbf{Q} \to \hat{C}(L)$.

A **homotopy** or **path** $\lambda : J \to \hat{C}(L)$ is determined by a linear filtered homomorphism $\bar{\lambda} : J \to L$. Denote the image of t_n by y_n and that of $t_n u$ by z_n. By the correspondence $Hom_{filt}(J, L) \to L \hat{\otimes} J^*$, we can represent $\bar{\lambda}$ by $\Sigma y_i t^i + dt \Sigma z_i t^i$. That λ is a dgc map is expressed by $d_L \bar{\lambda} + \bar{\lambda} d + \frac{1}{2}[\bar{\lambda}, \bar{\lambda}] = 0$ which translates to

$$d_L y_n + \frac{1}{2} \underset{i+j=n}{\Sigma} [y_i, y_j] = 0 \tag{3}$$

$$(n+1) y_{n+1} + d_L z_n + \underset{i+j=n}{\Sigma} [y_i, z_j] = 0. \tag{4}$$

Letting $\eta(t) = \Sigma y_n t^n$ and $\zeta(t) = \Sigma z_n t^n$ in $L[[t]]$, we have the differential equations

$$d_L \eta + \frac{1}{2}[\eta, \eta] = 0 \tag{5}$$

$$\frac{d\eta}{dt} + d_L \zeta + [\eta, \zeta] = 0. \tag{6}$$

The first equation says that $\eta(t)$ is a perturbation, while the second gives an action of $L^0[[t]]$ on the set of perturbations. To check that if $\eta(0)$ satisfies the MC equation, so does $\eta(t)$ for all t, proceed as follows: Let $u(t) = d_L\eta + \frac{1}{2}[\eta, \eta]$. Then

$$\frac{du}{dt} = dL\frac{d\eta}{dt} + [\frac{d\eta}{dt}, \eta] =$$

$$d_L(-d_L\zeta - [\eta, \zeta]) - [d_L\zeta + [\eta, \zeta], \eta] =$$

$$-d_L[\eta, \zeta] - [d_L\zeta, \eta] - [[\eta, \zeta], \eta] =$$

$$-[\zeta, d_L\eta] - [[\eta, \zeta], \eta],$$

using $[\eta, \zeta] = -[\zeta, \eta]$. By the Jacobi relation.

$$[\eta, \zeta], \eta] = -[[\eta, \eta], \zeta] + [\eta, [\zeta, \eta]],$$

so $\frac{1}{2}[[\eta, \eta], \zeta] = [\eta, \zeta], \eta]$. Thus

$$-[\zeta, d_L\eta] - [[\eta, \zeta], \eta] = -[\zeta, d_L\eta + \frac{1}{2}[\eta, \eta]].$$

In other words,

$$\frac{du}{dt} = -[\zeta, u].$$

In fact, this gives

$$u(t) = u(0)exp(-[\zeta,\ ]t).$$

If $\eta(0)$ satisfies the MC equation, i.e.$u(0) = d_L\eta(0) + \frac{1}{2}[\eta(0), \eta(0)] = 0$, then $u(t) = 0$ for all t (by uniqueness of solutions of ODE).

Later in **??** we will replace L by its homology with respect to d_L together with the L_∞-algebra structure transferred from the strict dg Lie algebra structure of L. The MC equation is correspondingly generalized to

$$dp + \sum \frac{1}{n!}[p, ..., p] = 0.$$

The second differential equation generalizes in a perhaps less obvious way:

$$\frac{d\eta}{dt} + d_L\zeta + \sum \frac{1}{n!}[\eta, ..., \eta, \zeta] = 0,$$

where there are n factors of η. These differential equations correspond to equations in (formal) power series. (Although the word *formal* is of necessity used in two different ways in this paper: in the sense of homotopy type and in the sense of power series; hopefully the context will make it clear which is intended.)

We will use the second differential equation to prove the main result from a homotopy point of view, but first we need to worry about the transition from the formal theory using

formal power series to the subtler results involving convergence. We are concerned with $L \subset Der\, L(\mathcal{H})$ or $Der\, A(L(\mathcal{H}))$ consisting of the weight decreasing derivations. L is complete with respect to the weight filtration, so $C(L)$ is with respect to the induced filtration and hence with respect to the $\otimes$–filtration.

IS THAT RIGHT? SO WE DON'T NEED TO COCOMPLETE??

Definition 4.7. *For a complete dg Lie algebra L with filtration F_p and a filtered dgcc C, a dgcc map $f : C \to C(L)$ is* **filtered** *if $\pi f : C \to L$ is filtration preserving. In particular, a homotopy $\lambda : J \to C(L)$ being* **filtered** *implies $\Sigma\pi\lambda(t_n u)$ and $\Sigma\pi\lambda(t_n u)$ are well–defined elements of L^1 and L^0 respectively.*

Definition 4.8. *A* **path component** *of $C(L)$ for a complete dg Lie algebra L is a filtered homotopy class of points: $\mathbf{Q} \to C(L)$.*

Remark. The Main Homotopy Theorem could be rephrased entirely in terms of "classifying twisting cochains" $\pi\chi(p) : \mathbf{Q} \to L$ and filtered homotopies thereof.

Motivation from topology, especially the generalization to the classification of fibrations §9, are better served by staying in the category of dgcc's.

In light of Theorem 4.3, we have that the Main Homotopy Theorem is equivalent to:

Main Homotopy Perturbation Theorem 4.9. *Two perturbations p and q represent the same augmented homotopy type if and only if the classifying maps $\chi(p), \chi(q) : \mathbf{Q} \to C(L)$ are homotopic as filtered maps of coalgebras.*

Here L is the weight decreasing subalgebra of $Der\, L(\mathcal{H})$. After proving the theorem, we investigate in the following section the possibility of replacing L by an equivalent dgL; some changes will be conceptually significant, others of computational importance.

4.2 Proof of the Main Homotopy Theorem

In this section, we use the basic deformation differential equations to provide the proof of the hard part of Theorem 4.1: homotopy implies equivalence. For the easy part, observe [20] that $(SZ, d + p)$ and $(SZ, d + q)$ have the same augmented homotopy type provided there is an automorphism $\mathbf{Q}$ of SZ of the form: Id plus "terms which decrease weight". The equivalence can be expressed directly in terms of p and q as elements of the Lie algebra $Der\, SZ$.

For any dg Lie algebra $(L = \oplus L_i, d)$, consider the adjoint action of L on L, i.e., $ad(x)(y) := [x, y]$. We can define an action of the universal enveloping algebra UL on L making L a UL module: for $u = x_1 \cdots x_n \in UL$ with $x_i \in L$, define $uy = [x_1, [x_2, [\cdots, [x_n, y] \cdots]]$ for $y \in L$. Provided L is complete with respect to the filtration $L \supset [L, L] \supset [L, [L, L]] \supset \dots,$ there is a sensible meaning to $(exp\, x)y = \Sigma \frac{x^n}{n!} y$ for $x \in L, y \in L$. In fact, $exp\, x$ acts as an automorphism of L. In particular, for the Lie algebra $L \subset Der\, SZ$ of weight decreasing derivations, L is complete with respect to the weight filtration and hence with respect to the action by UL.

For this L of weight decreasing derivations, we can similarly define $exp\, x$ as an automorphism of the algebra SZ for $x \in L$. Given a dga map and automorphism $\phi : (SZ, d+p) \to (SZ, d+q)$ of the form $Id +$ "terms which decrease weight", let $b = log(\phi - Id) \in L$ so that $exp\, b = \Sigma \frac{b^n}{n!} = \phi$. The equation $(d+q) \circ \phi = \phi \circ (d+p)$ can then be written

$$d + q = (expb) \circ (d+p) \circ (expb)^{-1}$$

which is the same as

$$d + q = (exp\, b)(d+p)$$

using the action of UL on L.

If we set $\zeta(t) = b$ and $\eta(t) = (exp\, tb)(d+p) - d$, the differential equations (5) and (6) are satisfied and $\eta(1) = q$, so $\chi(p)$ and $\chi(q)$ are filtered homotopic.

Given a homotopy, it is much harder to find ϕ because of the non-additivity of exp, but use of the differential equation viewpoint will allow us to succeed.

A homotopy $\lambda : J \to \mathcal{C}(L)$ gives a solution $(\eta(t), \zeta(t))$ of the differential equation. We will use that solution to solve the equivalence, i.e., find θ so that $d+q = (exp\, \theta)(d+p)$ where $q = \eta(1)$. We change point of view slightly and look for η given ζ. It is helpful first to write $\mu(t) = d + \eta(t)$, so that equation (6) becomes

$$d\mu(t)/dt + [\mu(t), \zeta(t)] = 0.$$

For the remainder of this section, $\mu(t)$ will have this meaning with $\mu(0) = d + p$. We will solve this equation formally, i.e., by power series, and then remark where appropriate on convergence.

Lemma 4.10. *If $\zeta(t) = z \in L$, then $\mu(t) = (exp\, t\zeta)\mu(0)$ is a solution of $d\mu(t)/dt + [\mu(t), \zeta] = 0$.*

Proof. $d\mu(t)/dt = \Sigma \frac{t^{n-1}}{(n-1)!}(ad^n\, z)\mu(0)$ while $[z, \mu(t)] = \Sigma \frac{t^n}{n!}[z, (ad^n z)\mu(0)]$. (Notice both begin with the term $[z, \mu(0)]$.) ∎

Similarly for $\zeta = t^k \phi$ with $\phi \in L$, we have that $\mu(t) = (exp\, \frac{t^{k+1}}{k+1}\, \phi)\mu(0)$ is a solution. We would like to handle a general homotopy, i.e., a general ζ, in a similar manner. We have to contend with the "additivity" of homotopy and the non–additivity (via Campbell–Hausdorf–Baker) of exp. For this purpose, we write $\mu(t) = u(t)\mu(0)$. In the case just studied, $u(t) = exp\, \frac{t^k}{k}\phi$ acts as an automorphism of L and is associated to $\zeta \in L[[t]]$ such that for any $\theta = d + \psi$ with $\psi \in L_1$, we have

$$\dot{u}\theta + [u\theta, \zeta] = 0.$$

We wish to preserve these attributes as we construct $\mu(t)$ for a more general $\zeta(t)$.

Consider the "additivity" of homotopy. If we have pairs (μ_1, ζ_1) and (μ_2, ζ_2) which are solutions of the differential equation

$$\dot{\mu} + [\mu, \zeta] = 0$$

with $\mu_1(0) = d + p$ and $\mu_2(0) = \mu_1(1) = d + q$, we wish to find a solution (μ_3, ζ_3) such that $\mu_3(0) = d + p$ and $\mu_3(1) = \mu_2(1)$. It is sufficient for our purposes to consider

$$\mu_1(t) = u_1(t)\mu_1(0), \tag{7}$$
$$\mu_2(t) = u_2(t)\mu_2(0) \tag{8}$$

where each $u_i(t)$ acts as an automorphism of $L[[t]]$ and, for any $\theta = d + \psi$ with $\psi \in L_1[[t]]$, we have

$$\dot{u}_i\theta + [u\theta, \zeta_i] = 0.$$

Lemma 4.11. *The pair $(\mu_3, \zeta_2 + u_2\zeta_1)$ is a solution for*

$$\mu_3(t) = u_2(t)u_1(t)\mu_1(0).$$

Proof. We compute

$$\begin{aligned}\dot{\mu}_3 &= \dot{u}_2 u_1 \mu_1(0) + u_2 \dot{u}_1 \mu_1(0)\\ &= -[u_2 u_1 \mu_1(0), \zeta_2] - u_2[u_1\mu(0), \zeta_1]\\ &= -[u_2 u_1 \mu_1(0), \zeta_2] - [u_2 u_1 \mu(0), u_2\zeta_1]\end{aligned}$$

as desired. □

Thus transitivity of homotopy corresponds to a sort of crossed additivity of the ζ's.

Since we have a solution for each $\zeta_k = t^k\phi$, the lemma can be applied inductively to show:

Corollary 4.12. *For any $\zeta(t) = \Sigma z_k t^k$ and given $d + p$, there is a (unique) formal solution $\eta(t)$ with $d + \eta(t)$ of the form*

$$\cdots exp\, t^n\theta_n \cdots exp\, t\theta_1(d + p), \quad \theta_i \in L_1.$$

Of course, the θ_i are in general not the z_i.

Now we address the question of convergence of $\mu(t)$. When L is complete and $J \to \mathcal{C}(L)$ is filtered, we have immediately that $\Sigma\ z_n$ is convergent. The deviation from additivity shows $n\theta_n$ differs from z_n by terms of still more negative weight; the finite product therefore converges also for $t \leq 1$.

Similarly, if $\phi_n = exp\,\phi_n$, then the sequence of automorphisms $\phi_n \cdots \phi_1$ converges to show that $d + q = d + \eta(1)$ is equivalent to $d + p$, which completes Theorems 1.3, 4.1.

5 Homotopy invariance of the space of homotopy types

One advantage of the homotopy theoretic point of view is that it suggests the homotopy invariance of the space M_L of augmented homotopy types with respect to changes in the dg Lie algebras used. We began with the filtered model (SZ, d) but could just as well have used the filtered model $A(L(\mathcal{H}))$.

Using the models (SZ, d) and $A(L(\mathcal{H}))$, we can consider perturbations of (SZ, d) as before, of $L(\mathcal{H})$ with respect to bracket length or of $A(L(\mathcal{H}))$ with respect to the grading induced from bracket length. Having perturbed $L(\mathcal{H})$ to $\bar{L}(\mathcal{H})$, it follows that $A(\bar{L}(\mathcal{H}))$ is a perturbation of $A(L(H))$. Since (SZ, d) is minimal, by Theorem **??** there is an induced map of *Der* SZ into *Der* $A(L(\mathcal{H}))$. In fact, regarding $(SZ, d) \to A(L(\mathcal{H}))$ as a model, it is not hard to see this map preserves both gradations, so a perturbation of SZ maps to a perturbation of $A(L(\mathcal{H}))$.

For dg Lie algebras, the *weight* of a derivation is the decrease in total degree plus the increase in the bracket length. For dgas, the *weight* of a derivation is, as before, the increase in total degree plus the decrease in resolution degree.

For any weighted dga A or dg Lie algebra L, let $W_A \subset Der\ A$ (respectively $W_L \subset Der\ L$) denote the subdg Lie algebra of weight decreasing (respectively increasing) derivations. Let $p \in W_{(SZ,d)}$ and $q \in W_{L(\mathcal{H})}$ have the same image in $W_{A(L(\mathcal{H}))}$, then the respective classifying maps give a commutative diagram:

$$\begin{array}{ccc} & & C(W_{(SZ,d)}) \\ & \nearrow & \downarrow \\ Q & \longrightarrow & C(W_{A(L(\mathcal{H}))}) \\ & \searrow & \uparrow \\ & & C(W_{L(\mathcal{H})}). \end{array}$$

At the end of 3, we saw that the maps of Ws induced homology isomorphisms. Thus the classifications are equivalent at the homological level. In fact, we get the same space of augmented homotopy types independent of the model used, as we now show in detail.

Definition 5.1. *For any dg Lie algebra (L, d_L), we define the* **incorporated** *dg Lie algebra $(L[d], ad\ d)$ by adjoining a single new generator also called d of degree one with the obvious relations: $[d, d] = 0, [d, \theta] = ad\ d_L(\theta)$.*

For any dg Lie algebra L, we define the variety $V_L \subset L[d]$ to be

$$\{p \in L^1 | (d + p)^2 = 0\}.$$

The variety is in fact in L where the defining equation might more appropriately be written

$$dp + 1/2[p, p] = 0,$$

called, at various times, the deformation equation, the integrability equation, the Master equation and now most commonly the Maurer-Cartan equation.

If L is complete with respect to the L^0filtration, the action

$$p \mapsto (exp\ b)(d+p) - d$$

makes sense in $L[d]$ for $p \in L^1, b \in L^0$. We define the quotient bf space M_L to be $V_L/expL^0$.

For complete dg Lie algebras, we have the notion of filtered homotopy of classifying maps and the Main Homotopy Theorem holds in that generality.

Theorem 5.2. *If L is an L^0-complete dg Lie algebra , then M_L is in one-to-one correspondence with the set of filtered homotopy classes of maps $Q \to \hat{C}(L)$.*

As for the homology invariance of M_L , we have:

Theorem 5.3. *Suppose $f : K \to L$ is a map of decreasingly filtered dg Lie algebras which are complete and bounded above in each degree. If f induces*
a monomorphism $H^2(K) \to H^2(L)$
an isomorphism $H^1(K) \to H^1(L)$ and
an epimorphism $H^0(K) \to H^0(L)$,
then $f$induces a one-to-one correspondence between M_K and M_L.

Definition 5.4. *Such an f will be called an* (homology) equivalence in degree 1*. The decreasingly filtered dg Lie algebras K and L will be said to be (homology) equivalent in degree 1.*

By *bounded above in each degree*, we mean there exists $N(i)$ such that $F^nL_i = 0$ for $n \geq N(i)$.

The proof will, in fact, show that f induces a homeomorphism between the appropriate quotient topologies on M_K and M_L.

Lemma 5.5. *Let $f : K \to L$ be a map of decreasingly filtered complexes, complete and bounded from above in each degree. Suppose i is an index such that $H^{i-1}(grf)$ is injective and $H^i(grf)$ is surjective, then the same is true respectively of $H^{i-1}(f)$ and $H^i(f)$.*

Proof. For the injectivity, consider the induced map of cohomology sequences for K and L respectively arising from $0 \to F^{n+l} \to F^n \to gr^nF \to 0$.

Consider $x \in H^{i-1}(F^n)$ such that $H(f)(x) = 0$. A standard diagram chase shows x comes from $y \in H^{i-1}(F^{n+l})$ and $H(f)(y) = 0$. Starting with $F^{N(i-1)} = K^{i-1}$, we have classes $x_k \in H^{i-1}(F^{N(i-1)+k})$ such that $x_k - x_{k+l} = 0$ in $H^{i-1}(F^{N(i-1)+k})$. Now for $x \in H^{i-1}(K)$ such that $H(f)(x) = 0$, we have

$$x = \sum_{k \geq 0} x_k - x_{k+l},$$

which makes sense since the filtration is complete. On the other hand, the right hand side has each term 0, so x is 0 and $H^{i-1}(f)$ is injective.

Similarly, for $x \in H^i(K) = H^i(F^{N(i)})$, there is $y \in H^i(F^{N(i)})$ such that $x - H(f)(y)$ comes from $H^i(F^{N(i)+1})$. By induction then, there are $y_k \in H^i(F^{N(i)+k})$ such that $x - \sum_{j=1}^{k-1} H(f)(y_j)$ comes from $H^i(F^{N(i+1)})$, hence, in the limit, $x = H(f)(\sum y_k)$. ☐

The following lemma is familiar in the ungraded case.

Lemma 5.6. *Suppose K is a complete dg Lie algebra . If $\theta = d + p \in d + K^1$ and $b \in K^0$ is of positive filtration such that $(exp\ b)\theta = \theta$, then $[b, \theta] = 0$.*

Proof. We will show that $[b, \theta]$ has arbitrarily high filtration and hence is zero. Suppose $[b, \theta] = c_n \in F^n K^1$. Then

$$exp(b)\theta = \theta + [b, \theta] + 1/2[b, [b, \theta]] + \cdots \tag{9}$$

$$= \theta + c_n + 1/2[b, c_n] + \cdots \tag{10}$$

If $(exp\ b)\theta = \theta$, then $c_n + 1/2[b, c_n] + \cdots = 0$, but b has positive filtration, so $1/2[b, c_n]$ and the further terms come from F^{n+1}, hence for the sum to be zero, c_n must come from F^{n+1}. Thus $[b, \theta]$ has arbitrarily high filtration and must be zero. ☐

Proof. **of Theorem 5** The map $f : K \to L$ induces a map of the spaces of perturbations $f : V_K \to V_L$.

$f : V_K \to V_L$ is **surjective**

Let $q \in L^1$ be a perturbation, i.e. $(d + q)^2 = 0$. Assume we have constructed $p \in K^1$ and $b \in L^0$ such that $(d + p)^2 = a_{n+l}$ of filtration $n + 1$ and $(exp\ b)(d + fp) = d + q + c_n$ with c_n of filtration $n \geq 1$. Squaring the second equation gives

$$((exp\ b)(d + fp))^2 = [d, c_n] + [q, c_n] + c_n^2$$

while applying$(exp\ b)^2 f$ to the first shows this is also $(exp\ b)^2 f a_{n+1}$. Since q is of filtration ≥ 1 and $n \geq 1$, this shows $[d, c_n]$ is of filtration $n + 1$, i.e., c_n is a cycle mod filtration $n + 1$ and hence there exists $r \in K^1$ of filtration $\geq n + 1$ and $c \in L^0$ such that

$$(exp(b + c))(d + fp + fr) \equiv d + q$$

modulo filtration $n+1$. Since $(d+q)^2 = 0$, we can further choose r so that $(d+p+r)^2$ is of filtration $n+2$, completing the induction. Thus, since K^1 and L^0 are complete, $f : V_K \to V_L$ is onto.

$M_K \to M_L$ is **injective**

Suppose we have perturbations $p, q \in K^1$ which are equal modulo $F^n K$ and such that $(exp\ b)(d + fp) \equiv d + fq$ for $b \in L^0$ of positive filtration. Write $q - p = a_n$ of filtration n, then in $L/F^{n+1}L$ we have

$$[b, d + fp] + fa_n,$$

so the isomorphism $H^1(K, d + p) \simeq H^1(L, d + fp)$ implies there is a $c \in K^0$ such that

$$[c, d+p] \equiv a_n \text{ modulo} F^{n+1}.$$

Now $(exp\ c)(+p) \equiv d+q$ modulo F^{n+1}.

On the other hand, $(exp\ b)(d+fp) \equiv d+fq$ implies $[b, d+fp] = fa_n + e_{n+1}$ with $e_{n+} \in F^{n+1}L$. Since q is a perturbation, we have $[d+p, a_n] \equiv 0$ modulo F^{n+2}, so e_{n+1} is a $(d+fp)$-cycle modulo F^{n+2}. Thus there is a $(d+p)$-cycle $z_{n+1} \in F^{n+l}K/F^{n+2}$ such that $e_{n+1} = fz + [g, d+fp]$ for some $g \in L^0/F^{n+2}$. Replace b by $b = b - fc - g$ and c by c' such that $[c', d+p] \equiv a_n - z_{n+l}$ modulo F^{n+2}. Consider $(exp\ b')(exp\ fc')(d+fp)$. Modulo F^{n+2}, this is

$$d + fp + [fc', d+fp] + [b', d+fp] \tag{11}$$
$$\equiv d + fp + fa_n fz_{n+1} + fa_n + e_{n+1} fa_n e_{n+1} + fz_{n+1} \tag{12}$$
$$\equiv d + fp + fa_n \tag{13}$$
$$\equiv d + fq. \tag{14}$$

The induction is complete. □

M_L as a **space**

As promised, we point out that if K and L are regarded as topological vector spaces with the topology given by the filtrations, then $f : K \to L$ is an open map and hence $V_K \to V_L$ is an open surjection or quotient map. Thus the quotient $M_K \to M_L$ is not only a bijection but a homeomorphism. Notice that if $H^2(f)$ is mono and $H^1(f)$ only onto, the first half of the proof goes through, i.e., $M_K \to M_L$ will still be onto. In particular, let $f : K \to L$ be defined by

$$K^i = 0, i \leq 0,$$
$$K^1 \text{is a complement to } dL^0 \subset L^1 (i.e., L^1 = dL^0 \oplus K^1),$$
$$K^i = L^i, i > 1.$$

Thus $H^i(f)$ is an isomorphism for $i > 0$, so $V_K = M_K \to M_L$ is onto.

Observe that in classifying homotopy types, we began with perturbations of the filtered model (SZ, d) but have also considered other models such as $A(L(\mathcal{H}))$. This led us to consider $K \subset Der\ L(H)$ and $L \subset Der\ A(L(\mathcal{H}))$. The homology isomorphism $sL(H)\sharp Der\ L(\mathcal{H}) \to Der\ A(L(\mathcal{H}))$ of Theorem 3.17 restricts to a homology isomorphism $K \to L$ since the weight decreasing derivations of $L(\mathcal{H})$ induce derivations of $A(L(\mathcal{H}))$ decreasing weight by the same amount and $sL(\mathcal{H})$ corresponds to derivations of $A(L(\mathcal{H}))$ which do not decrease weight. Finally, the natural map $Der\ C(L(\mathcal{H})) \to Der\ A(L(\mathcal{H}))$ is an isomorphism provided $L(\mathcal{H})$ is of finite type and concentrated in degrees < 0, as it is for $\mathcal{H}$ simply connected of finite type. (Recall $Der\ C(L(\mathcal{H}))$ consists of coderivations.)

6 Control by L_∞-algebras.

An essential ingredient of our work is the combination of the deformation theoretic aspect with a homotopy point of view. Indeed we adopted the philosophy, later promoted by Deligne [8] in response to Goldman and Millson [17] (see [18] for a history of that development), that any problem in deformation theory is "controlled" by a differential graded Lie algebra, unique up to quasi-isomorphism of dg Lie algebras . Neither the variety V nor the group G are unique, but the quotient $M = V/G$ is (up to appropriate isomorphism).

Implicit in the use of quasi-isomorphisms, even for strict dg Lie algebras, is the fact that L_∞-morphisms respect the deformation and moduli space functors.

6.1 Quasi-isomorphisms and homotopy inverses

Definition 6.1. *A morphism in a category of dg objects is a* quasi-isomorphism *if it induces an isomorphism of the respective cohomologies* as graded objects*. For dg objects over a field, quasi-isomorphisms are also known as* weak homotopy equivalences.

Indeed, for dg vector spaces, a quasi-isomorphism always admits a homotopy inverse in the category of dg vector spaces, (i.e. a chain homotopy inverse). That is, a morphism $f : A \to B$ is a homtopy equivalence means there exists a homotopy inverse, a morphism $g : B \to$A such that fg is homotopic to Id_B and gf is homotopic to Id_A. If f respects additional structure, such as that of a dg Lie algebra , the inverse g need not; however, it will respect that structure *up to homotopy* in a very strong sense, e.g. as an L_∞-morphism. Thus, even when a controlling dg Lie algebra is at hand, comparison of relevant dg Lie algebras is to be in terms of L_∞-morphisms. Hence, one should consider control more generally by L_∞-algebras. In terms of a dg Lie algebra L as we have been doing, an attractive candidate is its homology $H(L)$, not as the obvious dg Lie algebra with trivial d but rather with the more subtle L_∞-structure as transferred from L, e.g. via a Hodge decomposition of L [25]. This is why we introduced L_∞-algebras in our early drafts, although they had been implicit in Sullivan's models. Since such algebras are now well established in the literature, we recall just a few aspects of the theory. The following definition follows our cohomological convention; d is of degree 1. The original definition was homological, d of degree -1 and thus the operations are of degree $k-2$. Special cases of L_∞-algebras L occur with names such as *Lie n-algebras*. An important distinction exists according to bounds for L from above or below.

Definition 6.2. *An* L_∞**-algebra** *is a graded vector space* L *with a sequence* $[x_1, \ldots, x_k]$, $k > 0$ *of graded antisymmetric operations of degree* $2-k$*, such that for each* $n > 0$*, the* n*-Jacobi relation holds:*

$$\sum_{k=1}^{n} \sum_{\substack{i_1<\cdots<i_k;j_1<\cdots<j_{n-k} \\ \{i_1,\ldots,i_k\}\cup\{j_1,\ldots,j_{n-k}\}=\{1,\ldots,n\}}} (-1)^\epsilon \, [[x_{i_1}, \ldots, x_{i_k}], x_{j_1}, \ldots, x_{j_{n-k}}] = 0.$$

Here, the sign $(-1)^\epsilon$ equals the product of the sign $(-1)^\pi$ associated to the unshuffle as a permutation with the sign associated by the Koszul sign convention to the action of the permutation.

The operation $x \mapsto [x]$ makes the graded vector space L into a cochain complex, by the 1-Jacobi rule $[[x]] = 0$. Because of the special role played by the operation $[x]$, we denote it by d. An L_∞-algebra with $[x_1, \ldots, x_k] = 0$ for $k > 2$ is the same thing as a dg Lie algebra. Just as an ordinary Lie algebra can be captured by its Chevalley-Eilenberg chain complex, so too for L_∞-algebras by shifting the degrees. In terms of the graded symmetric operations

$$\ell_k(y_1, \ldots, y_k) = (-1)^{\sum_{i=1}^{k}(k-i+1)|y_i|}\, s^{-1}[sy_1, \ldots, sy_k]$$

of degree 1 on the graded vector space $s^{-1}L$, the generalized Jacobi relations simplify to become

$$\sum_{k=1}^{n} \sum_{\substack{i_1<\cdots<i_k, j_1<\cdots<j_{n-k} \\ \{i_1,\ldots,i_k\}\cup\{j_1,\ldots,j_{n-k}\}=\{1,\ldots,n\}}} (-1)^{\tilde{\epsilon}}\, \{\{y_{i_1}, \ldots, y_{i_k}\}, y_{j_1}, \ldots, y_{j_{n-k}}\} = 0,$$

where $(-1)^{\tilde{\epsilon}}$ is the sign associated by the Koszul sign convention to the action of π on the elements $(y_1, \ldots, y_n)$ of $s^{-1}L$. Moreover, the operations ℓ_k can be summed (formally or in the nilpotent case) to a single differential and coderivation on $C(L)$.

A quasi-isomorphism of L_∞-algebras is an L_∞-morphism which is a quasi-isomorphism of the underlying cochain complexes.

There are at least two sign conventions for the definition, which are usable cf. [34, 35, 16], just as there are for the definition of an A_∞-algebra, for which there is a 'geometric' explanation: a classifying space BG can be built from pieces $\Delta^n \times G^n$ or from $(I \times G)^n$.

6.2 L_∞-structure on $H(L)$

Since the dg Lie algebra algebras that are manifestly controlling the deformations of interest are huge, it is helpful to have at hand the induced L_∞-structure on $H(L)$. Even though $H(L)$ inherits a strict dg Lie algebra structure (with $d = 0$), there is in general a highly non-trivial L_∞-structure such that L and $H(L)$ are equivalent *as L_∞ algebras*. This *transfer of structure* began in the case of dg associative algebras with the work of Kadeishvili [28]. The definitive treatment in the Lie case (which is more subtle) is due to Huebschmann [25].

In terms of the $C(L)$ definition, almost everything we have done in earlier sections carries over. What changes most visibly are the form of the Maurer-Cartan equation and the differential equation governing the equivalence.

Once we had that L_∞-structure on $H(Der\mathcal{A})$, our 'perturbations' (also known as 'integrable elements') could be regarded as solutions of what is now called the ∞-MC equation:

$$\sum 1/n![y, ..., y, y] = 0.$$

Notice even though the differential on $H(L)$ is zero, we can still have a non-trivial L_∞ structure, i.e. suitably compatible multibrackets

$$[\ ,\dots,\ ] : H^{\otimes n} \to H.$$

Implicit (only) was the fact that the gauge equivalence of the original dg Lie algebra transfers to an ∞-action of H^0 on solutions of an ∞-MC equation; even though H^0 is a strict Lie algebra, it is the ∞-action that gives the gauge equivalence. Some researchers have missed this point and tried to use just the strict Lie algebra acting on H^1. By the way, it's not so obvious that the ∞-action gives an equivalence relation, as we shall see later.

7 The Miniversal Deformation

7.1 Introduction

In this chapter, our point of view moves from homotopy theory to algebraic geometry. The moduli space of rational homotopy types with given cohomology H (assumed here finite dimensional) has the form W/G, where W is a conical affine algebraic variety and G is an algebraic "gauge" group, or rather, groupoid, acting on W. Later we will treat W more precisely as a *scheme*, that is, a functor from algebras to sets or topological spaces. Unfortunately, W/G is rarely what is called a fine module space. First, W/G is usually not a variety - e.g. the ubiquitous presence of non-closed orbits in W prevents the points of W/G from being closed . Second, even when W/G is a variety, there may well be no total space $X \to W/G$ with rational homotopy space fibers. (X would be a differential graded scheme, represented algebraically by either an almost free dgca, free just as gca, over R, where $Spec\ R = W/G$ or by an almost free dgla over R.) We can deal with these difficulties by first considering the "moduli functor" M of augmented rational homotopy types.

For an augmented algebra A , with pointed affine scheme $Spec\ A = S$, this functor assigns to A or S the set of families $Y \to S$, together with an isomorphism of the special fiber with the formal space $X_{\mathcal{H}}$, all modulo gauge equivalence. We view $Y \to S$ as a "deformation" of $X_{\mathcal{H}}$, parameterized by S. This functor is not in general representable, as that would require a "universal" deformation $X \to W$ or fine moduli space. However, there is nevertheless a "miniversal" deformation $X \to W$ inducing every deformation $Y \to Spec\ S$ by a map $S \to W$. This map is not unique , but its tangent at the base point is. The (Zariski) tangent space to W at its base point is $H^1(L)$, where L is the dgla of nilpotent derivations of the minimal (algebra or Lie algebra) model of $\mathcal{H}$, as introduced previously. In the "unobstructed" case, W is the affine space $H^1(L)$, but in general, W will be a subvariety of $H^1(L)$ defined by m homogeneous polynomial functions of degree at least 2, where $m = dim\ H^2(L)$. The nilpotent Lie algebra $H^0(L)$ is the tangent space to a unipotent groupoid U , such that W/U is the (not fine) moduli space of augmented rational homotopy types with cohomology H. The groupoid G for unaugmented homotopy types is an extension of $Aut\ H$ by U.

The construction of X, W and U proceeds from the minimal model L' of the dgla of negative weight derivations of a minimal model of H. This L' is an L_∞ algebra which consists of $H(L)$, together with higher order brackets $[a, b, c] \in H^{(|a|+|b|+|c|-1)}$ etc. of all orders > 2 , and all the brackets together satisfy the usual generalized Jacobi identities [35, 16].

Then $W = MC(L)$ is the set of $a \in H^1(L)$ which satisfy the Maurer-Cartan integrability condition

$$1/2[a, a] + 1/3![a, a, a] + ... = 0$$

(the sum is finite), U is the groupoid attached to the L_∞ algebra L', and X is constructed from the derivation representation of L'.

We give precise definitions and basic details in the following sections.

7.2 Varieties and schemes

An affine k variety, embedded in the affine space k^n, is the set of n-tuples in k^n which satisfy a collection of polynomial equations from $P := k[x_1, ..., x_n]$. By contrast, an affine k-scheme $X = Spec\ P/I$ is given by the choice of an ideal $I \subset P$. Then , for each overfield j of k, we have the affine j- variety $X(j) \subset j^n$, of points in j^n which satisfy the polynomials of I. Thus, the equations $f = 0$ and $f^2 = 0$ define the same varieties, but different schemes. The ideal I is not determined by the variety $X(k)$, or even by the collection $\{X(j)\}$. However, if, for a variable k-algebra A, we let $X(A) \subset A^n$ denote the the set of points in A^n which satisfy the polynomials in I, then the functor $A \mapsto X(A)$ determines the ideal I or the scheme X. Indeed:

$$X(A) = Hom(P/I, A).$$

An affine scheme is called *reduced* if its ideal I equals its radical- that is, P/I contains no nilpotents except 0. Over an algebraically closed field, there is a one-to-one correspondence between reduced schemes and varieties - the two notions collapse to one .

A point p in the variety $X(k)$ is the same as an augmentation of the algebra $R = P/I$. For a point $p = a = (a_1, \cdots, a_n)$, the augmentation X_a is defined by evaluating at a all polynomials in $P := k[x_1, ..., x_n]$. The vector space

$$X_a(k[\epsilon]), \epsilon^2 = 0$$

of augmentation preserving points in $X(k[\epsilon])$ is called the *Zariski tangent space* to X at $p = a = (a_1, \cdots, a_n)$ and

$$X_a(k[\epsilon_n]), \epsilon_n^{n+1} = 0$$

is the set of n-th order jets to X at p.

In addition to being a functor from algebras to sets, $X = Spec\ R$ is also a local ringed space. The (scheme) points p of X are the prime ideals in R; the closed subsets of X are the primes containing J for J an ideal in R , so that the basic open sets are the subsets of the form $X_f = \{p : f \notin p\}$ for $f \in R$. The closed and basic open sets in this Zariski topology support subschemes *Spec* (R/J) and *Spec* $(R[1/f])$ respectively and the direct limit of the $R[1/f]$, for $f \notin p$, is the local ring R_p of X at p. If we let $f(p)$ denote the image of $f \in R/p \subset R_p/pR_p$, then the condition $f(p) \neq 0$ defines the open set X_f.

A general scheme is defined to be a local ringed space which is locally affine; morphisms of schemes are morphisms of local ringed spaces. We have $Hom(Spec\ A, Spec\ B) = Hom(B, A)$. As a scheme, X determines a variety $X(k) = Hom(Spec\ k, X)$.

The basic example of a non-affine scheme is projective space P^n, for which $P^n(k) = k^{n+1} - \{0\}$ modulo the action of $G_m = k - \{0\}$. Thus $P^n = \cup\ Spec\ k[x_0/x_i, ..., x_n/x_i]$ where i runs from 0 to n. A quasi-projective (resp. quasi-affine) scheme is a locally closed subscheme of projective (resp. affine) space.

Finally, a differential graded scheme is the common generalization of scheme and rational homotopy space which results from replacing algebras with flat differential graded algebras over algebras .

7.3 Versal deformations

In any representation $M = V/G$ of a moduli space, or rather functor, M, as a quotient of a scheme V by a group scheme G acting on V, we call V a **versal** scheme for M. In practice, V should be the base scheme of a family $X \to V$ which is a **versal deformation** of a special fiber X_0. Here are the precise definitions . Let X_0 be a rational homotopy space, (R, d) a filtered model of X_0 and B an augmented algebra. Denote the augmentation ideal of B by m_B.

Definition 7.1. *A* **deformation of** X_0*, parameterized by Spec* B*, is a family Spec* $C \to$ *Spec* B *where* $C = (R \otimes B, d + e)$ *and*

- *a)* $e \in Der^1(R, R \otimes m_B)$ *lowers filtration,*
- *b)* $(d + e)^2 = 0$*, viewed as a derivation of* $R \otimes B$*.*
- *c) Spec* C *corresponds to the rational homotopy space of Sullivan and of Getzler [63, 16], which is the geometric realization of the dgca* C*.*

Let $M(B)$ be the set of such deformations, modulo isomorphism. A deformation $X \to V$ will be *versal* if the induced map $V \to M$ of functors is surjective, with a little more called "smoothness ". We consider functors X of augmented algebras. For these we assume that $X(k) = \{0\}$ is one point.

Definition 7.2. *A map* $X \to Y$ *of functors (from augmented algebras to sets) is* smooth *if for each surjection*$B \to B'((m'_B)^2 = 0)$ *of augmented algebras , with* $(m'_B)^2 = 0$*, the induced map*

$$X(B) \to X(B') \times_{Y(B')} Y(B)$$

is surjective.

If we take B' to be the ground field k, we find that $X(B) \to Y(B)$ is surjective.

We pause to investigate the smoothness condition for maps in the context of formal geometry, where we replace schemes by formal schemes, augmented algebras by complete local algebras, polynomials by formal power series.

- a) A scheme is smooth (over *Spec* k) exactly when it is non-singular.
- b) A group scheme (in characteristic 0) is smooth.
- c) A principal bundle is smooth over its base.
- d) If a group scheme G acts on a scheme V, then the map $V \to V/G$ of functors is smooth.

In the presence of a G_m action ("weighting"), we do not need to pass to the formal setting. For example , a smooth conical scheme is a weighted affine space, i.e. the Spec of a weighted polynomial algebra. Recall that a formal rational homotopy space $X_{\mathcal{H}}$ has a G_m action, so that deformation data attached to it will also .

Definition 7.3. *A deformation $X \to V$ is* versal *if the induced map of functors $V \to M$ is smooth.*

If $V \to M$ is smooth, we will see that $M = V/G$, for suitable G. The definition of deformation of X_0 suggests that we can apply a construction to the dgla $L = Der_{-}(R)$ (for R a filtered model of X_0) to get a formally versal deformation of X_0. Indeed, we can attatch to L the *Maurer-Cartan scheme* $V = V_L$ whose points with values in B are given by

$$V(B) = \{e \in L^1 \otimes m_B | [d, e] + 1/2[e, e] = 0\}$$

We claim first that $V = Spec\ A$, where $A = H^0(A(L^+))$, provided L^1 has finite dimension. In fact, if we take $A = H^0(A(L^+))$ and B is any augmented algebra, $Hom(A(L), B) = V(B)$ so that

$$Hom(A, B) = Hom(A(L), B) = V(B).$$

Thus $V = Spec\ A = H^0(A(L^+))$. (Here Hom denotes augmentation preserving morphisms.) Next , the equality $Hom(A(L), A) = V(A)$ gives us a tautological $e \in V(A)$ and thus a differential $d + e \in Der_{-}(R \otimes A)$. This in turn yields a deformation $X = Spec\ C \to V$ with $C = (R \otimes A, d + e)$. From chapter **??**, we see that the induced map of functors $V \to M$ is surjective and that $V/G = M$, where $G = exp\ L^0$. By example d) above, $V \to M$ is formally smooth or $X \to V$ is a formally versal deformation of X_0. If $X_0 = X_{\mathcal{H}}$, then $X \to V$ is a (conical) versal deformation of $X_{\mathcal{H}}$.

The above (formally) versal deformation is certainly not unique. We may replace $X \to V$ by $X \times (X_0 \times S) \to V \times S$, where S is smooth, to get another. We can also change the model R of X_0, or replace $L = Der_{-}(R)$ by a model $L' \to Der_{-}(R)$. These last replacements provide an L' which is quasi-isomorphic to L, but as we will see in section 7.4, the corresponding deformations are related by a smooth factor as above.

Finally, we discuss infinitesimal criteria for versality. A functor X from augmented algebras to sets has a tangent vector space $TX = X(k[\epsilon]/\epsilon^2)$. For $X = Spec\ A$, we have $TX = (m_B/m_B^2)^*$, while for $X = V_L$, we have $TX = Z^1(L)$ and for $X = M_L, TX = H^1(L)$. There are also normal spaces: $NSpec\ A = (I/m_P I)^*$, where $A = P/I$ for P a polynomial algebra with I included in $(m_P)^2$. For $M = M_L$, we have $NM = H^2(L)$.

For any deformation $X \to V$, we have Kodaira-Spencer maps

$$t : TV \to TM$$

$$n : NV \to NM.$$

$X \to V$ is formally versal if and only if t is surjective and n is injective. A deformation is *miniversal* if it is versal and t is an isomorphism .

7.4 The miniversal deformation

We give here the construction of the conical miniversal deformation of the formal rational homotopy space $X_{\mathcal{H}}$. The construction of the formally miniversal deformation of a general

X_0 is similar. Let R be a weighted model of $\mathcal{H}$ and $L \to Der_{-}R$ a model of the negative weight derivations of R. (We will see that the construction is independent of these choices.) If L^1 has finite dimension, we get a conical versal deformation $X \to V$ of $X_{\mathcal{H}}$. There are two ways in which V is deficient as an approximation to a fine moduli space. First, it is not a homotopy invariant of L - different L's do not yield isomorphic deformations. (The versal deformation is a homotopy invariant of L^+, not L.) Second, the approximation can be improved - the tangent space to V is $Z^1(L)$, whose dimension is greater than or equal to that of $H^1(L)$, with equality when the deformation is miniversal or $d|L^0 = 0$. To achieve the latter, we can replace L with an L_∞ minimal model $L' \to L$ with $d' = 0$. That is, L' is $H(L)$ with higher order brackets $[h_1, \cdots, h_n] \in H(L)$ adjoined as in 6. We denote the resulting free dgca by $A(L')$. The Maurer-Cartan condition becomes

$$mc(x) := 1/2[x, x] + 1/3![x, x, x] + \cdots = 0.$$

The sum is finite when $H(L)$ is of finite type, hence nilpotent. This *mc* condition then specifies the miniversal scheme W and the map $L' \to Der_{-}(R)$ gives us the miniversal deformation $X \to W$.

There are two ways to construct the miniversal deformation, or minimal model of L, assuming the tangent soace $H^1(L)$ has fintie dimension. First we may ignore the Lie algebras and appeal to the general construction of Schlessinger [55] and then invoke weight and nilpotence conditions to get a conical family. Or one can solve the mc equation in L by succesive approximations to yield the minimal model (in degree 1) of L, or the minimal model (in degree 0) of $A(L)$. We give an outline of the latter approach, assuming $H^1(L)$ is finite dimensional.

Let $L = H(L) \oplus R \oplus dR$ be a decomposition of the complex L. Take homogeneous elements $h_1, \ldots, h_p \in Z^1(L)$ which induce a basis of $H^1(L)$ and let $t^1, \ldots, t^p$ be dual coordinates. Set $x_1 = \sum h_i t^i$. Then we can find $< h_i, h_j > \in R^1$ such that , in Ł,

$$mc(x_1) = -1/2 \sum d < h_i, h_j > t^i t^j + [h_i, h_j] t^i t^j,$$

where [,] is the naive bracket $H^1(L) \otimes H^1(L) \to H^2(L)$.

Set $x_2 = \sum < h_i, h_j > t^i t^j$. Then

$$mc(x_1 + x_2) = 1/2 \sum [h_i, h_j] t^i t^j - 1/3! \sum d < h_i, h_j, h_k > t^i t^j t^k + [h_i, h_j, h_k] t^i t^j t^k.$$

CHECK CAREFULLY is that right now?

Here $< h, k, l > \in R^1$ and $[h, k, l] \in H^2$ if $h, k, l \ inH^1$.

Set $x_3 = 1/3! \Sigma < h_i h_j, h_k > t^i t^j t^k$, so

$$mc(x_1 + x_2 + x_3) = 1/2 \sum [h_i, h_j] t^i t^j + \sum 1/3! [h_i, h_j, h_k] t^i t^j t^k + O(4)$$

.

Continuing we get

$$mc_L(x_1 + x_2 + \cdots) = mc_{H(L)}(x_1).$$

Notice that the right hand side above is zero in $H(L)øB$, where $B = H^0(A(L'^+))$ and $W = Spec\ B$ is the base of the miniversal deformation. The x_i give a map transforming B to $A = H^0(A(L^+))$. A systematic and generalized treatment of the above construction is given in Huebschmann and Stasheff [26]. We now compare miniversal and versal deformations. If $X \to Spec\ A$ is versal and $Y \to Spec\ B$ is miniversal, then, by definition of these terms, we get weighted algebra maps $u : A \to B$ and $v : B \to A$ respecting the total spaces X and Y. The composition uv is the identity on the cotangent space $m/m^2 (m = m_B)$ and is therefore surjective. But a surjective endomorphism of a Noetherian ring is an automorphism. Hence we can change v to get $uv = identity$. We claim now that we have $A = B \otimes C$, where C is a weighted polynomial algebra . If we denote the cokernel of the tangential injection $u^* : T(B) \to T(A)$ by U, we may let C be the polynomial algebra on the dual positively weighted vector space U^*.

Thus $TC = U$, and the map $A \to C/(C_+)^2$ induces the trivial deforamtion over $C/(C_+)^2$. If we extend this family to a trivial family over C, the map extends to a map $A \to C$ inducing the trivial family over C, by the smoothness property of versality for A. This, together with $u : A \to B$, gives us a map $A \to B \otimes C$ which is an isomorphism on tangent spaces and respects total spaces. For the backwards map, we have $v : B \to A$ and $C \to C/(C+)^2$ lifts to $C \to A$, since C is a polynomial algebra. Thus we have:

Theorem 7.4. *A versal weighted family is isomorphic to the product of a miniversal weighted family and a trivial weighted smooth family $X_{\mathcal{H}} \times Spec\ C \to Spec\ C$ with C a polynomial algebra.*

In particular, two miniversal deformations are isomorphic, but not canonically.

The miniversal family $X \to W$ is formally versal for each of its fibers. At each point p of the base, the deformation splits, formally, into the product of the miniversal deformation of the fiber and a trivial deformation over the orbit of $p \in W$.

Now we list sufficient conditions for the finiteness we need. Given H of finite type, let $\mathfrak{g}$ be the free dgla on the suspension of H^{+*}. Then $R = A(\mathfrak{g})$ models $\mathcal{H}$ and $L = Der_-(\mathfrak{g})$ is a model of $Der_-(A(\mathfrak{g}))$

- a) L will have finite type if $\mathcal{H}$ is simply connected of finite dimension.
- b) $H(L)$ will have finite type if if $\mathcal{H}$ is finitely generated in even degrees and the associated projective scheme $Proj\ H$ is smooth. (Under the assumptions on $\mathcal{H}$, we will have $Spec\ \mathcal{H}$ as the cone over $Proj\ \mathcal{H}$.)
- c) We conjecture that $H(L)$ will have finite type if $\mathcal{H}$ is Koszul.

Finally , we point out $W = Spec\ A$ is the cone over $U = Proj\ A$ and the family $X \to W$, i.e. $Spec\ C \to Spec\ A$, is the cone over the family $X' \to U$ where $X' = Proj\ C$. The projective family has the same fibers as $X \to W$ does, i.e. every space with cohomology $\mathcal{H}$, except that $X_{\mathcal{H}}$ is missing. But the base is now compact.

7.5 Gauge equivalence for nilpotent L_∞-algebras

For any nilpotent L_∞-algebra K, the variety $MC(K)$ of Maurer-Cartan elements in K is the set of elements $x \in K^1$ satisfying the Maurer-Cartan condition

$$[x] + 1/2[x,x] + 1/3![x,x,x] + \cdots = 0.$$

Here $[x] = dx$, and the sum is finite by nilpotence, cf. [16]. The scheme V_K underlying the varierty $MC(K)$ is defined by the equation $V_K(B) = MC(K\text{ø}m_B)$ or $V_K = Spec\, H^0(A(K^+))$. If L is the minimal model of K, i.e. $d = 0 \in L$, then the Maurer-Cartan scheme of L isgiven by the equations

$$1/2[x,x] + 1/3![x,x,x] + \cdots = 0$$

for $x \in L^1$. Then $W = V_L$ is what is called the *miniversal scheme* of K - it is a homotopy invariant of K.

In particular, let K denote the tangent Lie algebra consisting of the negative weight derivations of either an algebra or Lie algebra model of $\mathcal{H}$. and let L be a minimal L_∞model ($d = 0$) of K. We assume that Łis concentrated in degrees 0, 1, 2 and has finite type. We then have the miniversal deformation $X \to W$ of $X_{\mathcal{H}}$, where $W = Spec\ A$ for $A = H^0(A(L^+))$. In case L is a dgla (third and higher order brackets as 0), the equivalence relation is given by the action of the unipotent group $G = exp(L^0)$ on the pure quadratic variety $MC(L)$ associated to W.

Here we have the setting for the Deligne groupoid attached to L. (A groupoid is a small category such that all morphims are invertible. The objects here form $MC(L)$ and the morphisms are given by the action of G.) Replacing L by $L\text{ø}B$, we obtain the Deligne stack (functor from algebras to groupoids) and the quotient of W by the stack is the *moduli functor* $\mathcal{M}$. For general L, this equivalence relation on W will have the form $U \rightrightarrows W$ for a scheme $U \rightrightarrows W$, where both relative tangent spaces identify to $H^0(L)$, together with an associative "Campbell-Hausdorff " law of composition $U \times_W U \to U$ which guarantees the transitivity of U, c.f. Schlessinger [55].

Thus we again have a stack in which the objects over B are given by $W(B)$ and the morphisms by $U(B)$. If we choose U minimal, the two smooth maps $U \rightrightarrows W$ each have relative tangent space $L^0 = H^0(L)$, which is a nilpotent Lie algebra, and U will be unipotent. The replacement of L by another model of the tangent algebra will result in an equivalent, but larger, stack with unknown tangent spaces.

A description of U in terms of L is given by Getzler [16]. He starts with the simplicial set (actually, variety associated to a simplicial scheme) $MC_\bullet$. whose n-simplices are given by $MC_n = MC(L\text{ø}\Omega_n)$, where $\Omega_n = k[t_0, ..., t_n, dt_0, ..., dt_n]/(t_0 + ... + t_n)$. According to Fukaya, Oh, Ohta , and Ono , [FOOO], we get an equivalence relation $R = MC_1$ on $MC(L) = MC_0$. which unfolds as follows : $x_0 \in MC(L)$ is equivalent to $x_1 \in MC(L)$ if and only if there is an $x = a(t) + b(t)dt \in L[t, dt]$ with $a(t) \in MC(L[t]), b(t) \in L^0[t]$ with $a(0) = x_0, a(1) = x_1$ and $da/dt = 1/2[a,b] + 1/3![a,b,b] + \cdots$. But this variety R is too large: it is not even a groupoid. It has the relative tangent space $L^0[t]$ instead of L^0. Getzler obtains a groupoid by replacing $MC_\bullet$ by a much smaller sub-simplicial set (scheme) $\gamma_\bullet(L)$ given by the vanishing

of the "Dupont gauge " $s : MC(L)_\bullet \to MC_{\bullet -1}(L)$. (This will restrict $b(t)$ above to be constant.) This $\gamma_\bullet$ is the nerve of a groupoid. The stack U is obtained by replacing L by $L\o B$; it determines the moduli functor M.

Theorem 7.5. *The following stacks are equivalent:*

a) The stack whose objects over B are families $X \to Spec\ B$ having fiber cohomology $\mathcal{H}$ and whose morphisms are isomorphisms of families.

b) The stack U associated to the the minimal model of the tangent algebra of $\mathcal{H}$.

Remark 7.6. *i) The stack U determines the moduli functor as $\mathcal{M} = W/U = \pi_0(U)$. For $W = Spec\ A$, the equivalence relation R on $W(B) = Hom(A, B)$ is also given by homotopy of maps $A(L) \to B$.*

ii) If L is replaced by a quasi-isomorphic dgla K, then U is replaced by the stack determined by the action of $G = exp(K^0)$ on the Maurer-Cartan scheme $V = Spec\ H^0(A(K^+))$. Here we have a simpler action, but the actors G and V have no direct description in terms of the tangent cohomology $H(K) = L$.

7.6 Summary

Let $\mathcal{H}$ be a simply connected graded commutative algebra of finite type, $X_\mathcal{H}$ the formal rational homotopy space with cohomology $\mathcal{H}, \mathfrak{g}$ the free dgla generated by $s(\mathcal{H}^+)^*$ with differential dual to the multiplication in $\mathcal{H}$, i.e. the Quillen free dgla model of $\mathcal{H}$ or $X_\mathcal{H}$, and $R = A(\mathfrak{g})$ the corresponding free dgca model of $\mathcal{H}$ or $X_\mathcal{H}$. Let $Der_-(\mathfrak{g})$ be the dgla of weight decreasing derivations of $\mathfrak{g}$ (which models $Der_-(R)$ and $L \to Der_-(\mathfrak{g})$ the minimal $(d_L = 0)$ L_∞-model of both $Der_-(\mathfrak{g})$ and $Der_-(R)$. Let $W = Spec\ A$, for $A = H^0(A(L^+))$, be the Maurer-Cartan scheme of L and $e \in MC(Der_-(R)\o A)$ (i.e.$(d+e)^2 = 0$) be the corresponding classifying derivation. Assume $dim\ L^i = dim\ H^i(Der_-)$ is finite for $i = 0, 1, 2$.

Theorem 7.7. *There is a "miniversal deformation" $X \to W$ of $X_\mathcal{H}$ with the following properties:*

a) $W = Spec\ A$ is a conical affine scheme and $X = Spec\ (R\o A, d + e)$ is a dg scheme,

b) Each fiber of $X \to W$ is a rational homotopy space with cohomology $\mathcal{H}$ and every such space occurs as a fiber, up to isomorphism ,

c) The fiber over the vertex 0 in W is $X_\mathcal{H}$,

d) No fiber X_p for $p \neq 0$ is isomorphic to $X_\mathcal{H}$,

e) W is defined by $m = dim\ L^2$ weighted homogeneous polynomial equations, without linear terms, in $n = dim\ L^1$ variables,

f) The equivalence relation on W governing the duplication of the X_p, or the passage from W to moduli, is given by a conical affine scheme U with two smooth projections $U \rightrightarrows W$; the fiber dimensions are $p = dimL^0$, In fact W determines the objects, U the morphisms in a unipotent groupoid, or rather stack, U. The composition of morphisms is determined by a Campbell-Hausdorff map $c : U \times_W U \to U$,

g) At each point $p \in W$, the family $X \to W$ splits, formally, into the product of the miniversal deformation of the fiber X_p and a trivial family $X_{\mathcal{H}} \times S \to S$ where the smooth base S is the orbit of $p \in W$ under U.

The miniversal family $m : X \to W$ satisfies certain mapping properties. Any family $n : Y \to T$ of rational homotopy spaces with cohomology $\mathcal{H}$, deforming $X_{\mathcal{H}}$, is isomorphic to the pullback of m by a map of $T \to W$. If n is versal, i.e. contains all homotopy types as fibers, then n splits, formally, into the product of m and a trivial family over a smooth base, as in g). (This applies in particular to the MC family attached to an L_∞-algebra quasi-isomorphic to L).

Item e) says that the Zariski tangent space to W at 0 is L^1. This and versality uniquely determine m up to isomorphism. The relative tangent space to U over W is L^0, as W is the MC scheme attached to L and U is the MC scheme attached to $L' = L[t, dt]$, defined by the conditions $u = u_1(t) + u_0(t)dt$ with $u_i \in L^i[t]$ with u_0 constant.

If L is a dgla (third and higher order brackets vanish), then W is a pure quadratic cone and U is the action of the unipotent group $exp(L^0)$ on W.

If $L^2 = 0$, then W is smooth, i.e. W is the weighted affine space L^1. More generally, W is smooth when all brackets of r elements in L^1 vanish for $r > 0$.

If $L^1 = 0$, then $W = 0 = M$; $\mathcal{H}$ is intrinsically formal.

If $L^0 = 0$, then $U = 0$ and $M = W$.

The Campbell-Hausdorff map c is determined by the brackets $[x_1, \cdots, x_r, u, v] \in L^0$ with $x_i \in L^1$ and $u, v \in L^0$. If these vanish for $r > 0$, then U degenerates into the action of the group $exp(L^0)$.

If $X \to S$ is formally versal and the tangent map is an isomorphism, the family is*formally miniversal*. Such a family is then unique up to (formal) isomorphism. Under suitable nilpotence conditions, below, the adjective ”formal” may be dropped.

8 Examples and computations

Although some of our results are of independent theoretical interest, we are concerned primarily with reducing the problem of classification to manageable computational proportions. One advantage of the miniversal variety is that it allows us to read off easy consequences for the classification from conditions on $H(L)$.

For the remainder of this section, let (SZ, d) be the minimal model for a gca $\mathcal{H}$ of finite type and let $L_{\mathcal{H}} \subset Der\, L^c(\mathcal{H})$ be the corresponding dg Lie algebra of weight decreasing derivations, which is appropriate for classifying homotopy type. The following theorems follow from 7.7 and following remarks.

Theorem 8.1. *If $H^1(L_{\mathcal{H}}) = 0$, then $\mathcal{H}$ is intrinsically formal, i.e., no perturbation of (SZ, d) has a different homotopy type; $M_{\mathcal{H}}$ is a point.*

Theorem 8.2. *If $H^0(L_{\mathcal{H}}) = 0$, then $M_{\mathcal{H}}$ is the quotient of the miniversal variety by $Aut\,\mathcal{H}$.*

Theorem 8.3. *If $H^2(L_{\mathcal{H}}) = 0$, then the miniversal variety is $H^1(L_{\mathcal{H}})$.*

Theorem 8.4. *If $L_{\mathcal{H}}$ is formal in degree 1 (in the sense of* **??***), then $M_{\mathcal{H}}$ is the quotient of a pure quadratic variety by the group of outer automorphisms of (SZ, d) (cf. §7.4).*

The following examples give very simple ways in which these conditions arise. Let Der^n_k denotes derivations which raise top(ological) degree by n and decreases the weight = top degree plus resolution degree by k. For $Der^n_k L^c(\mathcal{H})$, this specializes as follows: $Der\, L^c(\mathcal{H})$ can be identified with $Hom\,(L^c(\mathcal{H}), \mathcal{H})$ and hence with a subspace of $Hom\,(T^c(\mathcal{H}), \mathcal{H})$ where $T^c(\mathcal{H})$ is the tensor coalgebra. Then each $\theta_k \in Der^n_k$ corresponds to an element of $Hom\,(\mathcal{H}^{\otimes k+p+1}, \mathcal{H})$ which lowers weight by k. In particular, θ_k of top degree 1 and weight $-k$ can be identified with an element of $Hom\,(\bar{\mathcal{H}}^{\otimes k+2}, \mathcal{H})$ which lowers the internal $\mathcal{H}$-degree by k (e.g., $d = m : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$ preserves degree). Thus examples of the theorems above arise because of gaps in $Der^n_k L^c(\mathcal{H})$ for $n = 0, 1$, or 2.

8.1 Shallow spaces

By a **shallow space**, we mean one whose cohomological dimension is a small multiple of its connectivity.

Theorem 8.5. *[13, 48, 20] If $\mathcal{H}^i = 0$ for $i < n > 1$ and $i \geq 3n - 1$, then $\mathcal{H}$ is intrinsically formal, i.e., $M_{\mathcal{H}}$ consists of one point.*

From our point of view or many others, this is trivial. We have $\bar{\mathcal{H}}^{\otimes k+2} = 0$ up to degree $(k + 2)n$, so that $Image\ \theta_k$ lies in degree at least $(k + 2)n - k$ where $\mathcal{H}$ is zero for $k \geq 1$. A simple example is $\mathcal{H} = H(S^n \vee S^n \vee S^{2n+1})$ for $n > 2$.

Theorem 8.6. *If $\mathcal{H}^i = 0$ for $i < n$ and $i \geq 4n - 2$, then the space of homotopy types $M_{\mathcal{H}}$ is $H^1(L)/Aut\,\mathcal{H}$ [13, 12].*

Proof. Now $L^1_{\mathcal{H}} = L^1_1$, i.e., θ_1 may be non–zero but $\theta_k = 0$ for $k \geq 2$. Similarly $L^2_{\mathcal{H}} = 0$. Thus, $W = V_{L_{\mathcal{H}}} = Z^1(L)$. Consider $L_{\mathcal{H}}$ and its action. The brackets have image in dimension at least $3n-2$, thus in computing $(exp\,\phi)(d+\theta)$ the terms quadratic in ϕ lie in $\mathcal{H}^i$ for $i \geq 4n-2$ and hence are zero. Thus, $(exp\,\phi)(d+\theta)$ reduces to $(1 + ad\,\phi)(d+\theta)$. The mixed terms $[\phi, \theta]$ again lie in dimension at least $4n-2$ and are also zero, so that $(exp\,\phi)(d+\theta)$ is just $d + \theta + [\phi, d]$. Therefore, $W_{L_{\mathcal{H}}}/exp\ L_{\mathcal{H}}$ is just $H^1(L_{\mathcal{H}})$. $\square$

Here a simple example is $\mathcal{H} = H(S^2 \vee S^2 \vee S^5)$ [20] §6.6. Let the generators be x_2, y_2, z_5. We have $L^2 = 0$ for the same dimensional reasons, so $W_L = V_L = H^1(L)$ which is $\mathbf{Q}^2$. Finally, $Aut\,\mathcal{H} = GL(2) \times GL(1)$ acts on $H^1(L)$ so as to give two orbits: $(0,0)$ and the rest. The space $M_{\mathcal{H}}$ is

$$\odot \ \cdot$$

meaning the non-Hausdorff two-point space with one open point and one closed. For later use, we will also want to represent this as $\cdot \to \cdot$, meaning one orbit is a limit point of the other.

If $\mathcal{H}^i = 0$ for $i < n$ and $i \geq 5n-2$, then $W = V_L = Z^1(L)$ still, but now the action of L may be quadratic and much more subtle. We will return to this shortly, but first let us consider the problem of invariants for distinguishing homotopy types.

8.2 Cell structures and Massey products

We have mentioned that $Der\ L(\mathcal{H})$ can be identified with $Hom\,(\mathcal{H}^*, L(\mathcal{H}))$. This permits an interpretation in terms of attaching maps which is particularly simple in case the formal space is a wedge of spheres $X = \bigvee S^{n_i}$. The rational homotopy groups $\pi_*(\Omega X) \otimes \mathbf{Q}$ are then isomorphic to $L(H(X))$ [23]. In terms of the obvious basis for $\mathcal{H}$, the restriction of a perturbation θ to $\mathcal{H}_{n_i}$ can be described as iterated Whitehead products which are the attaching maps for the cells e^{n_i} in the perturbed space.

In more detail, here is what's going on: attaching a cell by an ordinary Whitehead product $[S^p, S^q]$ means the cell carries the product cohomology class. Massey (and Uehara) [72, 40] introduced Massey products in order to detect cells attached by iterated Whitehead products such as $[S^p, [S^q, S^r]]$. If we identify a perturbation θ_k with a homomorphism $\theta_k : H^{\otimes k+2} \to H$, this suggestion of a $(k+2)$–fold Massey product can be made more precise as follows: Consider the term θ of least weight k in the perturbation. By induction, we assume all j-fold Massey products are identically zero for $3 \leq j < k+2$. Now a $(k+2)$–fold Massey product would be defined on a certain subset $M_{k+2} \subset H^{\otimes k+2}$, namely, the kernel of $\Sigma(-1)^j(1 \otimes \cdots \otimes m \otimes \dots 1)$ which is to say

$$x_0 \otimes \cdots \otimes x_{k+1} \in M_{k+2} \qquad \text{iff} \sum_{j-0}^{k} (-1)^j x_0 \otimes \cdots \otimes x_j x_{j+1} \otimes \cdots \otimes x_{k+1} = 0.$$

We can then define $\langle x_0, \dots, x_{k+1} \rangle$ as the coset of $\theta(x_0 \otimes \cdots \otimes x_{k+1})$ in H modulo $x_0 H + H x_{n+1}$.

Moreover, if $\theta = [d, \phi]$ for some $\phi \in L$, then $\langle x_0, \dots, x_{k+1}\rangle$ will be the zero coset because

$$\begin{aligned}\theta(x_0 \otimes \cdots \otimes x_{k+1}) =& x_0\phi(x_1 \otimes \cdots \otimes x_{k+1})\\ &\pm \Sigma(-1)^j \phi(x_0 \otimes \cdots \otimes x_j x_{j+1} \otimes \cdots \otimes_{k+1})\\ &\pm \phi(x_0 \otimes \cdots \otimes x_k)x_{k+1},\end{aligned}$$

the latter sum being zero on M_{k+2}. Notice ϕ makes precise "uniform vanishing".

The first example of "continuous moduli", i.e., of a one–parameter family of homotopy types, was mentioned to us by John Morgan (cf. [47, 13]). Let $\mathcal{H} = H(S^3 \vee S^3 \vee S^{12})$, so that the attaching map α is in $\pi_{11}(S^3 \vee S^3) \otimes \mathbf{Q}$ which is of dimension 6, while $Aut\,\mathcal{H} = GL(2) \times GL(1)$ is of dimension 5. Alternatively, the space of 5–fold Massey products $\mathcal{H}^{\otimes 5} \to \mathcal{H}$ is of dimension 6 and so distinguishes at least a 1–parameter family.

The Massey product interpretation is particularly helpful when only one term θ_k is involved. All of the examples of Halperin and Stasheff can be rephrased significantly in this form. To do so, we use the following:

Notation: Fix a basis for $\mathcal{H}$. For x in that basis and $y \in L(\mathcal{H})$, denote by $y\partial x$ the derivation which takes x to y (think of ∂x as $\partial/\partial x$) and sends the complement of x in the basis to zero.

8.3 Moderately shallow spaces

Returning to the range of $\mathcal{H}^i = 0, i < n$ and $i \geq 4n - 2$, consider Example 6.5 of [20], i.e., $\mathcal{H} = H((S^2 \vee S^2) \times S^3)$ with generators x_1, x_2, x_3. Again L^1 is all of weight -1; any θ_1 is a linear combination:

$$\begin{aligned}\mu_1[x_1,[x_1,x_2]]\partial x_1x_3 \;+&\; \mu_2[x_1,[x_1,x_2]]\partial x_2x_3\\ &+ \sigma_1[x_2,[x_1,x_2]]\partial x_1x_3 + \sigma_2[x_2,[x_1,x_2]]\partial x_2x_3.\end{aligned}$$

As for L, it has basis $[x_1, x_j]\partial x_3$ for $1 \leq i \leq j \leq 2$. Computing d_L, it is easy to see that $H^1(L)$ has basis: $[x_1,[x_1,x_2]]\partial x_1x_3 = -[x_2,[x_1,x_2]]\partial x_2x_3$. The action of $Aut\,\mathcal{H}$ again gives two orbits: $\mu_1 = \sigma_2$ and $\mu_1 \neq \sigma_2$. In terms of the spaces, we have respectively $(S^2 \vee S^2) \times S^3$ and $S^2 \vee S^3 \vee S^2 \vee S^3 \cup e^5 \cup e^5$ where one e^5 is attached by the usual Whitehead product and the other e^5 is attached by the usual Whitehead product plus a non–zero iterated Whitehead product.

Notice the individual Massey products $\langle x_1, x_1, x_2\rangle$ and $\langle x_1, x_2, x_2\rangle$ are all zero modulo indeterminacy (i.e., $x_1\mathcal{H}^3 + \mathcal{H}^3x_2$), but the classification of homotopy types reflects the "uniform" behavior of all Massey products. For example, changing the choice of bounding cochain for x_1x_2 changes $\langle x_1, x_1, x_2\rangle$ by x_1x_3 and simultaneously changes $\langle x_1, x_2, x_2\rangle$ by x_2x_3, accounting for the dichotomy between $\mu_1 = \sigma_2$ and $\mu_1 \neq \sigma_2$. The language of Massey products is thus suggestive but rather imprecise for the classification we seek.

Our machinery reveals that the superficially similar $\mathcal{H} = H((S^3 \vee S^3) \times S^5)$ behaves quite differently. There is only one basic element in L^1, namely $\phi = [x_1, x_2]]\partial x_5$, with again

$[d, \phi] = [x_1, [x_1, x_2]]\partial x_1 x_5 + [x_2, [x_1, x_2]]\partial x_2 x_5$, so $V_L / exp\, L \cong \mathbf{Q}^3$. If we choose as basic

$$\begin{aligned} p &= [x_1, [x_1, x_2]]\partial x_2 x_5 \\ q &= [x_2, [x_1, x_2]]\partial x_1 x_5 \\ r &= 1/2[x_1, [x_1, x_2]]\partial x_1 x_5 - 1/2[x_2, [x_1, x_2]]\partial x_2 x_5 \end{aligned}$$

then $GL(2, \mathbf{Q}) = Aut\, \mathcal{H}^3$ acts by the representation $sym\, 2$, the second symmetric power, that is, as on the space of quadratic forms in two variables. Since $Aut\, \mathcal{H}^5 = \mathbf{Q}^*$ further identifies any form with its non–zero multiples, the rank and discriminant (with values in $\mathbf{Q}^*/(\mathbf{Q}^*)2$) are a complete set of invariants. Thus there are countably many objects parameterized by $\{0\} \cup \mathbf{Q}/(\mathbf{Q}^*)^2$; in more detail, we have $\mathbf{Q}^*/(\mathbf{Q}^*)^2 \to 0 \to 0$, meaning one zero is a limit point of the other which is a limit point of each of the other points (orbits) in $\mathbf{Q}^*/(\mathbf{Q}^*)2$. Schematically we have

$$\begin{array}{ccc} \searrow & \downarrow & \swarrow \\ \longrightarrow & \cdot & \longleftarrow \\ & \downarrow & \end{array}$$

This can be seen most clearly by using $AutH$ to choose a representative of an orbit to have form

$$\begin{aligned} x_2 + dt_2, d \neq 0 \text{ (rank 2)} &\quad \text{or} \\ x_2 \text{ (rank 1)} &\quad \text{or} \\ 0 \text{ (rank 0).} & \end{aligned}$$

8.4 More moderately shallow spaces

Now consider $H^i = 0$ for $i < n$ and $i \geq 5n - 2$. We find $V_L = Z^1(L)$, but there may be a non–trivial action of L. Of course for this to happen, we must have $H^i \neq 0$ for at least three values of i, e.g., $H(X)$ for $X = S^3 \vee S^3 \vee S^5 \vee S^{10}$. Spaces with this cohomology are of the form $S^3 \vee S^3 \vee S^5 \cup e^{10}$. We have $V_L = Z^1(L) = L_1$ with basis

$$\begin{aligned} [x_i, [x_j, x_5]]\partial x_{10} &\qquad \text{for } L_1^1, \\ [x_i, [x_j, [x_1, x_2]]]\partial x_{10} &\qquad \text{with } i \geq j \text{ for } L_2^1. \end{aligned}$$

Thus L_1^1 corresponds to the space of bilinear forms (Massey products) on H^3:

$$\langle\ ,\ , x_5\rangle : H^3 \otimes H^3 \to H^{10} = \mathbf{Q}$$

and thus decomposes into symmetric and antisymmetric parts. On the other hand, L has basis $[x_1, x_2]\partial x_5$ and acts nontrivially on L_1^1 except for the antisymmetric part spanned by

$$[x_1, [x_2, x_5]]\partial x_{10} - [x_2, [x_1, x_5]]\partial x_{10} = [[x_1, x_2], x_5]\partial x_{10}.$$

(The *exp* L action corresponds to a one–parameter family of maps of the bouquet to itself which are the identity in cohomology but map S^5 nontrivially into $S^3 \vee S^3$.)

Now L_1^1 is isomorphic over $SL(2, \mathbf{Q}) = Aut\, H^3$ to the space of symmetric bilinear forms on H^3. If we represent $L^1 = L_2^1 \otimes L_1^1$ as triples (u, v, w) with u, w symmetric and v anti–symmetric, then $exp\, L$ maps (u, v, w) to $(u, v, tu+w)$. We have $Aut\ H^5 = \mathbf{Q}^*$ and $Aut\ H^{10} = \mathbf{Q}^*$ acting independently on L^1. If we look at the open set in L^1 where $v \neq 0$, we find the discriminant of u is a modulus. (In fact, even over the complex numbers, it is a nontrivial invariant on the quotient which can be represented as the $SL(2, \mathbb{C})$–quotient of $\{(u, \tilde{w}) | u \in \mathrm{sym}^2, \tilde{w} \in P^2(\mathbb{C}), \mathrm{discriminant}\,(\tilde{w}) = 0\}$.) The rational decomposition of the degenerate orbits proceeds as before.

8.5 Obstructions

We now turn to examples in which we do not have $\theta_1^2 = 0$ automatically and W_L may very well not be an affine space. One of the simplest examples of this phenomenon is $X = S^3 \vee S^3 \vee S^8 \vee S^{13}$ (we must have $\mathcal{H}^i \neq 0$ for at least three values of i); cf. [13, 12]. In particular, if

$$\theta_1 = [x_1, [x_1, x_2]]\partial y + [x_2, [x_1, y]]\partial z,$$

then $[\theta_1, \theta_1]$ does not represent the zero class in $H^2(L) = L^2$. Thus, $d+\theta_1$ can *not* be extended to $d + \theta_1 + \theta_2$ so that $(d + \theta_1 + \theta_2)^2 = 0$. We refer to θ_1 as an **obstructed** pre–perturbation or **obstructed infinitesimal** perturbation.

In terms of cells, this means we cannot attach both e^8 to realize $\langle x_1, x_1, x_2\rangle$ **and** attach e^{13} to realize $\langle x_2, x_1, x_8\rangle$.

The computations are essentially the same as those we now present for the miniversal scheme for homotopy types with cohomology

$$\mathcal{H} = H(S^k \vee \cdots \vee S^k \vee S^{3k-1} \vee S^{5k-2})$$

where $k > 1$ is an integer and there are $r > 1$ $k-$spheres. This is the simplest type of example with obstructed deformations of the formal homotopy type with cohomology $\mathcal{H}$; that is, the miniversal scheme W is singular. In fact, W_{red} is a "fan", the union of two linear varieties A and B meeting in the origin, corresponding to the two 'quanta' making up L^1. However, the scheme W is not reduced: If a_i and b_j are coordinates in A and B, then each product a_ib_j is nilpotent modulo the Maurer-Cartan ideal I, but few are in I.

Let $x_1, \ldots, x_r$ of degree $1-k$ be a basis of the suspended dual of $\mathcal{H}^k$ and let y and z play similar roles for the other cohomology in $\mathcal{H}$. The free graded Lie algebra generated by x_i, y, z with $d = 0$ is thus the Quillen model Q for $\mathcal{H}$ and the sub-Lie algebra of *Der* Q consisting of negatively weighted derivations is the controlling Lie algebra L for augmented homotopy

types with cohomology ∵ The subspace L^1 is the direct sum of subspaces A and B where A has a Hall basis consisting of the derivations $[x_i,[x_j,y]]\partial z$ (of weight -1) and the derivations $[x_k,[x_l,x_m]]\partial y, k \geq l < m$ (of weight -1) form a basis for B.

Thus $dim\ A = r^2$ and $dim\ B = 2\binom{r+1}{3}$ if k is odd . From now on, for convenience, we assume k is *odd*.

The bracket of the indicated basis elements is $[x_i,[x_j,[x_k,[x_l,x_m]]]]\partial z \in L^2$. If these are expressed as linear combinations of a basis of L^2, the transposed linear combinations exhibit generators for the Maurer-Cartan ideal I.

Let us analyze the situation in more detail. We have:

$$L^1 = L^1_1 \text{ with Hall basis } \begin{cases} \alpha_{ij} = [x_i,[x_j,y]]\partial z. \\ \beta_{klm} = [x_k,[x_l,x_m]]\partial y \end{cases}$$

with $k \geq l < m$. All brackets in $[L^1, L^1]$ are zero except

$$[\alpha_{ij}, \beta_{klm}] = [x_i,[x_j,[x_k,[x_l,x_m]]]]\partial z.$$

which we will denote γ_{ijklm}. It is not hard to see that the bracket map from the tensor product of A and B to L^2 is surjective, expressing things in terms of a Hall basis. The dimension of the space of quadratic generators I_2 of I is the same as $dim\ L^2 = dim\ F_5(r) \backsim r^5/5$. ($F_5(r)$ denotes the space of fifth order brackets in the free Lie algebra on r variables.)

The word γ_{ijklm} in Hall form is a *simple* word in L^2 , as opposed to the *compound* word $[[x_i,x_j],[x_k,[x_l,x_m]]]\partial z$, which we denote $\gamma_{ij|klm}$. A basis of L^2 is given by γ_{ijklm} together with $\gamma_{ij|klm}$ for $i \geq j, k \geq l < m$.

The duals of these Hall basis elements we will denote $a_{ij}, b_{klm}, c_{ijklm}$ and $c_{ij|klm}$ and will also use the products $a_{ij}b_{klm}$,

Finally, each of these symbols has a *content* $s = (s_1, s_2, \ldots, s_r)$, where s_i is the number of times i occurs in the symbol and has a *partition* $p = (p_1, p_2 \ldots, p_r)$ obtained by rearranging the s_i in descending order. Both these sequences add under bracketing or multiplying. We use a descending induction on the lexicographic order of the partition p attached to c_{ijklm} to prove that the latter is nilpotent. A key point is that the square of a product $a_{ij}b_{klm}$ is divisible by a product of higher partition mod I.

To illustrate these notions, we examine the situation when $r = 2$. We begin by expressing the brackets $[\alpha_{ij}, \beta_{klm}] = \gamma_{ijklm}$ in terms of the Hall basis in L^2, using the Jacobi relation repeatedly.

In the following table for content (3,2), the columns are headed by Hall basis elements of L^2; reading down yields a partial basis for I_2 , consisting of quadrics which generate the Maurer-Cartan ideal I. Further tables for other content (2,3), (4,1) and (1,4) yield a full set of generators for the Maurer-Cartan ideal I by the same procedure.

We organize the tables according to content .

(3,2)	21112	12|112
12112	1	1
21112	1	0
11212	1	1

In the above table, the rows give the expression of $[\alpha_{ij}, \beta_{klm}]$ in terms of the Hall basis of L^2. The columns then give the expansion of the dual basis elements as linear combinations of the products $a_{ij}b_{klm}$. Thus the first row says that $[\alpha_{12}, \beta_{112}] = \gamma_{2112} + \gamma_{12|112}$, etc. By reading down the columns, the first column says that the dual γ^*_{2112} is expressed as the quadric

- (1) $a_{12}b_{112} + a_{21}b_{112} + a_{11}b_{212} = s_{12}b_{112} + a_{11}b_{212}$ where $s_{12} := a_{12} + a_{21}$.

The second column gives a quadric

- (2) $a_{12}b_{112} + a_{11}b_{212}$.

Similarly, for the other contents $(2, 3), (4, 1), (1, 4)$, we get quadrics

- (3) $s_{12}b_{212} + a_{22}b_{112}$
- (4) $a_{12}b_{212}$
- (5) $a_{11}b_{112}$
- (6) $a_{22}b_{212}$.

These six quadrics generate the Maurer Cartan Ideal I. Multiplying (2) by a_{11}, we get

$$a_{11}a_{11}b_{212} \equiv -a_{12}a_{11}b_{112} \mod I$$

$$\equiv -a_{12} \cdot 0 \mod I.$$

Then $(a_{11}b_{212})^2 \equiv 0$, so $a_{11}b_{212}$ is nilpotent mod I.

Similarly $a_{22}b_{112}$ and $a_{21}b_{212}$ have square 0 mod I.

The other 3 quadrics, (4), (5), (6), are in I. (In addition to the above equations of the form $a^2b \equiv 0 \bmod I$, one also has a system of the form $ab^2 \equiv 0 \bmod I$.)

We note that the above procedure shows each product a^2b to be divisible by a product of higher partition (4) or (5), This is essentially the induction step.

To set up the induction step in the general setting, we outline what can be determined about the form of the quadrics in I_2. To each Hall word w of order 5 in the free Lie algebra on $x_1, \ldots, x_r$, there is associated a quadric q_w , a linear combination of the basic quadrics $a_{ij}b_{klm}$; the q_w form a basis of I_2. What are the restrictions on the word $u = ijklm,\ k \geq l < m$ such that $a_{ij}b_{klm}$ appear in q_w with non-zero coefficient ?

First, the content of w and u must be the same. It is not hard to see that it suffices to work under the assumption that this content is $(1, 1, 1, 1, 1, 0, 0, \ldots, 0)$, so that w and u are permutations of 12345. (For, if not, we take a suitable order preserving function f and transform q_w into $q_{f(w)}$.) There are 4 simple Hall words, 20 compound Hall words and 40 words $u = ijklm$ as above, so one gets a 40×24 matrix for which it is straightforward to fill in the rows with entries 0,1, or -1 for the Hall decomposition of u. Only the top half $(i < j)$ matters; the bottom is the same except for the subtraction of a 20×20 identity matrix. Here are the essential features, which generalize readily to other situations.

(1) Suppose that w is a simple Hall word, $w = pqrst$ with $p \geq q \geq r \geq s < t$. Then the product $u = ijklm$ ($k \geq l < m$) appears in q_w exactly when the contents $c(w) = c(u)$ and $l = t$ or $m = t$. The same applies to $a_{ij}b_{klm}$. Thus q_w is a sum of terms $s_{ij}b_{klm}$ (where $s_{ij} = a_{ij} + a_{ji}$).

(2) Suppose instead that w is a compound Hall word, $w = pq|rst$ with $p < q, r \geq s < t$. Then the summands of q_w are $a_{pq}b_{rst}, s_{pr}v_{qst}, s_{rs}v_{pqt}$. Here v_{xyz} stands for some linear combination of the two 3-letter Hall words spelled with x, y, z; e.g. v_{123} is a linear combination of b_{213} and b_{312}.

Lemma 8.7. Induction *Let $a = a_{ij}$ and $a' = a_{i'j'}$ be coordinates in A with different content so that $\{i, j\} \neq \{i', j'\}$, and b and b' are coordinates in B. Suppose ab and $a'b'$ have the same content. Then one of $a'b$ or ab' has higher partition than ab does.*

Proof. By reordering the indices if necessary, we may assume that the partition of ab equals its content. Suppose $c(a') > c(a)$. Then $c(a'b). = c(ab) = p(ab)$ so $p(a'b) \geq c(a'b')$ and hence $p(a'b) \geq c(a'b) > c(ab) = p(ab)$ as desired. The case $c(a') < c(a)$. is similar. $\square$

Theorem 8.8. *Let I be the Maurer Cartan Ideal for deformations of the homotopy type of $X = (S^k)^{\vee r} \vee S^{3k-1} \vee S^{5k-2}$, k odd, $r > 1$. Then $L^1 = H^1(L) = A \oplus B$, where A and B consist of unobstructed deformations and have coordinates a_{ij} and $b_{klm}, 1 \leq i, j, k, l, m \leq r, k \geq l < m$. Further, each product $a_{ij}b_{klm}$ is nilpotent modulo I.*

Corollary 8.9. *The miniversal variety W_{red} for deformations of the homotopy type of X decomposes as $W_{red} = A \vee B$ with an isolated singularity at the origin.*

Proof. To prove the theorem, we use descending induction on the partition $p(ab)$. When $p(ab) = (4, 1, 0, \ldots, 0)$ is as high as possible, then $ab = a_{ii}b_{iij}$ or $a_{ii}b_{iji} = q_w$ (where $w = iiiij$ or $w = iiiji$) is in I.

Suppose the theorem true for all products ab of partition $> p$ and consider ab of partition p and fixed content c. Give $q = sb$ where, for example, $s = s_{ij} = a_{ij} + a_{ji}$, is symmetric, then there is a simple Hall word w such that q_w contains q while for every other quadric $q = s'b'$ appearing in w we have $c(s) \neq c(s')$. By the Induction Lemma, qq' is divisible by a quadric of higher partition; thus qq' is nilpotent mod I. Since the square q^2 is minus the sum of such products mod I, we see q^2, and hence $q = sb$, is nilpotent mod I.

To prove that a product $q = ab$ is nilpotent, we note that $sb = 2ab$ when $a = a_{ii}$ is nilpotent, so ab is nilpontent mod I in this case. Thus we may assume $a = a_{ij}$ with $i < j$ and $b = b_{klm}$. Take w to be the compound word $w = ij|klm$. By (2) above, $q_w = ab+$ terms of the form $s'b'$. The latter are nilpotent mod I so ab is also. $\square$

Now consider the moduli space associated to X. Since L^1 has weight -1 and L^0 has negative weight, the action of the latter on the former is trivial. There remains to consider the action of $Aut\ \mathcal{H}$ on W where here $Aut\ \mathcal{H} = GL(r) \times G_m \times G_m$, where $G_m = GL(1)$. The number of continuous moduli for the action on B is $\binom{s+1}{2}$ when $r = 2s$ is even. We now analyze the entire moduli space $A/Aut\ \mathcal{H}$ when $r = 2$.

Let V be the span of x_1, x_2 so that $B \cong V \otimes V$ is a 4-plane on which $GL(2)$ acts as on bilinear forms on V and the action of G_m is by scalar multiplication. Taking normal forms for each point in the moduli space and letting $a \to b$ indicate that b is in the closure of a, we have:

$$\begin{array}{ccccc} & & (x_1+dx_2,1) & & \\ & r\swarrow & \downarrow & \searrow & \\ (x_1+dx_2,0) & \to & (x_2,0) & \leftarrow & (x_2,1) \\ & & \downarrow & & \\ & & (0,0) & & \end{array}$$

In $(x_1 + dx_2, 1)$, the continuous modulus is denoted by d, but in $(x_1 + dx_2, 0)$, it is determined only mod G_m^2. i.e. the forms $(x_2 + dy_2, 0)$ must be identified modulo squares: $d \sim \lambda^2 d$.

The bilinear form may be recovered from

$$< \quad , \quad , \ H^{3k-1} >: H^k \otimes H^k \to H^{5k-2}.$$

As for B, the order 3 part of the free Lie algebra on the x_i, the full analysis in terms of Geometric Invariant Theory is unknown past $r = 2$, where it is almost trivial: $\cdot \to \cdot$.

The obstructions can be interpreted in a very straightforward way, but with a perhaps surprising result. Any perturbation of $\mathcal{H}$ corresponds to a rational space of the form $S^k \vee S^k \cup e^{3k-1} \cup e^{13}$. The obstructions tell us that the deformations are either $S^k \vee S^k \vee S^{3k-1} \cup e^{5k-2}$ or $S^k \vee S^k \vee S^{5k-2} \cup e^{3K-1}$.

For the extreme case $k = 2$ of the previous example (i.e. $X = \vee^r S^2 \vee S^5 \vee S^8$), we have a non-trivial action of L^0 on L^1. Besides the previous subspaces of L^1, namely $A = \{x^3 \partial y\}, B = \{x^2 y \partial z\}$ of weight -1, we have an additional subspace $C = \{x^6 \partial z\}$ of weight -4. For L^0, there are two subspaces $D = \{x^4 \partial y\}$ and $E = \{x^3 y \partial z\}$ of weight -3 and $F = \{x^7 \partial z\}$ of weight -6. We find that $W_{red} = C \times (A \vee B)$ with $[B, D] = C = [A, E]$ with all other brackets between L^0 and L^1 being 0.

8.6 More complicated obstructions

Here we present the simplest obstruction for a bouquet such that W has an *non-linear* irreducible component V. In fact, V will have equations of the form

$$u_p v_q - u_q v_p, \quad 1 \leq p, q \leq c$$

(and some matrix generalization of this). Thus V is the cone over a "Segre" manifold $\mathbf{P}^{c-1} \times \mathbf{P}^1$ (or generalization thereof).

The obstructions arise in the setting $L^1 = A \oplus B$ again, where A consists of derivations of the form $\alpha = x^3 y \partial z$ and B consists of derivations of the form $\beta = x^3 \partial y$. These are realized

by a bouquet of spheres with $\mathcal{H} = \mathcal{H}^0 + \mathcal{H}^k + \mathcal{H}^{3k-1} + \mathcal{H}^{6k-2}$ where x, y, z run though a basis of the shifted dual of $\mathcal{H}^+$. We outline the construction.

We decompose A according to Hall type, i, j, k being the indices of the x_is :

- $A_1 : ijky\partial z, \quad i \geq j \geq k,$
- $A_2 : (ij)(ky)\partial z, \quad i < j,$
- $A_3 : yijk\partial z, \quad i \geq j < k.$

We also have $B : ijk\partial y, \quad i \geq j < k$. We set $c = dim\ A_3 = dim\ B$.

Let p denote the simple word ijk and q denote lmn and take

$$\alpha := \sum_{p=ijk} a_p[y, p]\partial z \in A_3$$

with $k \geq l < m$ and

$$\beta := \sum b_q q\partial y \in B,$$

then $[\alpha, \beta] = \sum_{p<q}(a_p b_q - a_q b_p)[p, q]\partial z$.

As the $[p, q]\partial z$ are linearly independent in L^2, we conclude that the scheme $\mathbf{P}^{c-1} \times \mathbf{P}^1$ defined by the equations $a_p b_q - a_q b_p = 0$ lies in W_{red}. Proceeding as before, we find that $W_{red} = V \cup L$ where L is a linear variety meeting V in another linear variety. This result has some interesting generalizations if $dim\ \mathcal{H}^{3k-1} = s > 1$, while $\mathcal{H}^k$ and $\mathcal{H}^{6k-2}$ have dimensions r and 1 as before. We let c denote the dimension of the the space of simple words $ijk,\ i \geq j < k$ in r variables and proceed as above. We obtain a variety $V \subset W$ consisting of pairs of matrices M, N such that M is a $c \times s$ matrix, N is an $s \times c$ matrix and MN is symmetric. $GL(s)$ acts on V and the quotient is the variety of $c \times c$ matrices with $rank \leq s$. We conjecture that V is a component of W.

On the other hand, the obstructions above can be avoided by adding S^{10} to $S^3 \vee S^3 \vee S^8$ and then attaching e^{13} so as to realize $x_2 x_{10}$. Then the class of $[\theta_1, \theta_1]$ is zero in $H^2(L)$, namely, $[\theta_1, \theta_1] = [d, \theta_2]$ for

$$\theta_2 = [x_1, [x_1, [x_1, x_2]]]\partial x_{10}.$$

8.7 Other computations

Clearly, further results demand computational perseverance and/or machine implementation by symbolic manipulation and/or attention to spaces of intrinsic interest.

Tanré [66] has studied stunted infinite dimensional complex projective spaces $\mathbb{C}P^\infty_n = \mathbb{C}P^\infty/\mathbb{C}P^n$. Initial work on machine implementation has been carried out by Umble and has led, for example, to the classification of rational homotopy types X having $H^*(X) = H^*(\mathbb{C}P^n \vee \mathbb{C}P^{n+k})$ for k in a range [73]. At the next level of complexity, he and Lupton [38] have classified rational homotopy types X having $H^*(X) = H^*(\mathbb{C}P^n/\mathbb{C}P^k)$ for all n and k:

For further results, both computational and theoretical, consult the extensive bibliography created by Félix building on an earlier one by Bartik.

9 Classification of rational fibre spaces

The construction of a rational homopy model for a classifying space for fibrations with given fibre was sketched briefly by Sullivan [63]. Our treatment, in which we pay particular attention to the notion of equivalence of fibrations, is parallel to our classification of homotopy types. Indeed, the natural generalization of the classification by path components of $C(L)$ provides a classification in terms of homotopy classes of maps $[C, C(L)]$ of a dgc coalgebra into $C(L)$ of an appropriate dg Lie algebra L. However, the comparison with the topology is more subtle; the appropriate $C(L)$ has terms in positive and negative degrees, because L does, unlike the chains on a space.

Because of the convenience of Sullivan's algebra models of a space and because of the applications to classical algebra, we present this section largely in terms of dgca's and in particular use $A(L)$ rather than $C(L)$ to classify. The price of course is the need to keep track of finiteness conditions.

Tanré carried out the classification in terms of Quillen models [64] (Corollaire VII.4. (4)). with slightly more restrictive hypotheses in terms of connectivity.

9.1 Algebraic model of a fibration

The algebraic model of a fibration is a twisted tensor product. For motivation, consider topological fibrations, i.e., maps of spaces

$$F \to E \xrightarrow{p} B$$

such that $p^{-1}(*) = F$ and p satisfies the homotopy lifting property. We have not only the corresponding maps of dgca's

$$A^*(B) \to A^*(E) \to A^*(F)$$

but $A^*(E)$ is an $A^*(B)$–algebra and, assuming $A^*(B)$and$A^*(F)$ of finite type, there is an $A^*(B)$–derivation D on $A^*(B) \otimes A^*(F)$ and an equivalence

$$\begin{tikzcd} & A^*(E) \arrow[dr] & \\ A^*(B) \arrow[ur] \arrow[dr] & & A^*(F) \\ & (A^*(B) \otimes A^*(F), D)\,. \arrow[ur] & \end{tikzcd}$$

To put this it our algebraic setting, let F and B be dgca's (concentrated in non–negative degrees) with B **augmented**.

Definition 9.1. *A sequence*

$$B \to E \to F$$

of dgca's such that F is isomorphic to the quotient $E/\bar{B}E$ (where $\bar{B}$ is the kernel of the augmentation $B \to \mathbf{Q}$) is an **F fibration over B** *if it is equivalent to one which as graded vector spaces is of the form*

$$B \xrightarrow{i} B \otimes F \xrightarrow{p} F$$

with i being the inclusion $b \to b \otimes 1$ and p the projection induced by the augmentation.

. Two such fibrations $B \to E_i \to F$ are **strongly equivalent** *if there is a commutative diagram*

$$\begin{array}{ccccc} B & \to & E_1 \to & & F \\ id \downarrow & & \downarrow & & \downarrow id \\ B & \to & E_2 \to & & F \end{array}$$

(It follows by a Serre spectral sequence argument that $H(E_1) \cong H(E_2)$.)

Both the algebra structure and the differential may be twisted, but if we *assume that F is free as a cga*, then it follows that E is strongly equivalent to

$$B \xrightarrow{i} B \otimes F \xrightarrow{p} F$$

with the $\otimes$–algebra structure. The differential in $E = B \otimes F$ then has the form

$$d_{\otimes} + \tau,$$

where

$$d_{\otimes} = d_B \otimes +1 \otimes d_F.$$

The "twisting term" τ lies in $Der^1(F, \bar{B} \otimes F)$, the set of derivations of F into the F-module $\bar{B} \otimes F$. This is the sub-dg Lie algebra of $Der(B \otimes F)$ consisting of those derivations of $B \otimes F$ which vanish on B and reduce to 0 on F via the augmentation.

Assuming B is connected, τ does not increase the F–degree so we regard τ as a perturbation of $d_{\otimes}$ on $B \otimes F$ with respect to the filtration by F degree. The twisting term must satisfy the integrability conditions:

$$(d + \tau)^2 = 0 \ \text{ or } \ [d, \tau] + \frac{1}{2}[\tau, \tau] = 0. \tag{15}$$

To obtain strong equivalence classes of fibrations, we must now factor out the action of automorphisms θ of $B \otimes F$ which are the identity on B and reduce to the identity on F via augmentation. Assuming B is connected, then $\theta - 1$ must take F to $\bar{B} \otimes F$ and therefore lowers F degree, so that $\phi = \log \theta = \log(1 + \theta - 1)$ exists; thus $\theta = exp\,(\phi)$ for ϕ in $Der^0(F, B \otimes F)$. If we set $L = L(B, F) = Der\,(F, \bar{B} \otimes F)$, then for B connected, we may apply the considerations of §5 to the dg Lie algebra L, because the action of L on L^1 is complete in the filtration induced by F–degree. The variety $V_L = \{\tau \in L^1 | (d + \tau)^2 = 0\}$ is defined with an action of $exp\,L$ as before.

Theorem 9.2. *For B connected, F free of finite type and $L = Der\,(F, \bar{B} \otimes F)$, there is a one–to–one correspondence between the points of the quotient $M_L = V_L/exp\,L$ and the strong equivalence classes of F fibrations over B.*

Notice that if F is of finite type, we may identify $Der\,(F, F \otimes \bar{B})$ with $(Der\ F)\hat{\otimes}\bar{B}$, i.e.,

$$Der^k(F, F \otimes \bar{B}) \cong \prod_n Der^{k-n}(F) \otimes \bar{B}^n.$$

Corollary 9.3. *If $H^1(L) = \Pi H^{1-n}(Der\,F) \otimes H^n(B)(n > 0)$ is 0, then every fibration is trivial. If $H^2(L) = 0$, then every "infinitesimal fibration" $[\tau] \in H^1(L)$ comes from an actual fibration (i.e., there is $\tau' \in [\tau]$ satisfying integrability, i.e. the Maurer-Cartan equation).*

We now proceed to simplify L without changing $H^1(L)$, along the lines suggested by our classification of homtopy types.

First consider $F = (SY, \delta)$, not necessarily minimal. Combining Theorems 3.12 and 3.17, we can replace $Der\,F$ by $D = sLH \sharp Der\ LH$ where LH is the free Lie algebra on the positive homology of F provided with a suitable differential. If $dim\,H(F)$ is finite, D will have finite type, so we apply the A–construction to obtain $A(D)$. Let $[A(D), B]$ denote the set of augmented homotopy classes of dgca maps (cf. Definition 4.2).

9.2 Classification theorem

The proof of Theorem 1.3 carries over to

Theorem 9.4. *There is a canonical bijection between $M_{D\hat{\otimes}\bar{B}}$ and $[A(D), B]$, that is, $A(D)$ classifies fibrations in the homotopy category.*

However, $A(D)$ now has negative terms and can hardly serve as the model of a space. To reflect the topology more accurately, we first truncate D.

Definition 9.5. *For a Z graded complex D (with differential of degree +1), we define the n^{th} truncation of D to be the complex whose component in degree k is*

$$\begin{aligned} D^k \cap ker\,d \quad & if \;\; k = n, \\ D^k \quad & if \;\; k < n, \\ 0 \quad & if \;\; k > n. \end{aligned}$$

We designate the truncations for $n = 0$ and $n = -1$ by D_c and D_s respectively (connected and simply connected truncations).

Theorem 9.6. *Let F be a free dgca and $D = sLH \sharp Der\ LH$ as above. If B is a connected (respectively simply connected) dgca, there is a one–to–one correspondence between classes of F fibrations over B and augmented homotopy classes of dga maps $[A(D_c), B]$ (resp. $[A(D_s), B]$).*

Thus $A(D_c)$ corresponds to a classifying space $B\ Aut\,\mathcal{F}$ where $Aut\,\mathcal{F}$ is the topological monoid of self–homotopy equivalence of the space $\mathcal{F}$. Similarly, $A(D_s)$ corresponds to the simply connected covering of this space, otherwise known as the classifying space for the sub–monoid $S\,Aut\,F$ of homotopy equivalences homotopic to the identity.

Proof. (Connected Case) $D \to Der\, F$ is a cohomology isomorphism. For $K = D_c \hat{\otimes} \bar{B}$ and $L = Der\, F \hat{\otimes} \bar{B}$, it is easy to check that $K \to L$ is a *homotopy equivalence in degree one* in the sense of **??**, so that M_K is homeomorphic to M_L. By 9.6, $[A(D_c), B]$ is isomorphic to M_K, which corresponds to fibration classes by 9.4. □

If we set $\mathfrak{g} = D_c/D_s$, so that $H(\mathfrak{g}) = H^0(D)$, then the exact sequence

$$0 \to D_s \to D_c \to \mathfrak{g} \to 0$$

corresponds to a fibration

$$B\, S\, Aut\, F \to B\, Aut\, F \to K(G, 1)$$

where G is the group of homotopy classes of homotopy equivalences of F, otherwise known as "outer automorphisms" [63].

Proof. (Simply connected case) Since $A(\ )$ is free, the comparison of maps $A(D_c) \to B$ and $A(D_s) \to B$ can be studied in terms of the twisting cochains of the duals $D_c^* \to B$ and $D_s^* \to B$. Since D_c^* and D_s^* differ only in degrees 0 and 1, if B is simply connected, the twisting cochains above are the same, as are the homotopies. □

The k^{th} rational homotopy groups ($k > 1$) of $A(D_s)$ and $A(D_c)$ are the same, (namely, $H^{-k+1}(Der\ F)$), but the cohomology groups are not.

In the examples below, we will need the following:

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If D_c denotes the connected (non positive) truncation of the Tangent Algebra of a formal space, then we have

Theorem 9.7. *$H(A(D_c))$ is concentrated in weight 0 .*

Proof. We take D to be the outer derivations of the free Lie algebra $LH = \mathcal{L}(x_1, ...x_n)$. Here x_i has topological degree t_i, i.e. $t_i = 1 - s_i$, where the s_i are the degrees in a basis of H). In addition to the topological degree t in LH, we have a bracket or resolution degree r, and a weight $w = t - r$. In D_c, we have $t \leq 0, r \geq 0, w \leq 0$. The differential in D_c, arising from the multiplication in H, has degrees $1, 1, 0$ with respect to t, r, w. Thus the differential in D_c or its cochain algebra $A(D_c)$ preserves weight .

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The key point is that the weight grading in D_c (or D) is induced by bracketing with the derivation

$$\theta = \Sigma(t_i - 1)x_i \partial x_i$$

which lies in the weight 0 part of D_c. That is, if ϕ has weight w, then $[\theta, \phi] = w\phi$. Fuks in [**?**] refers to w as an "internal grading " of D_c. It is easy to deduce the dgla version of Theorem 1.5.2 in that text and conclude that $H(A(D_c))$ is concentrated in weight 0, as desired. □

Note that in $A(D_c)$ we have $t \geq 1, r \leq 1, w = t - r \geq 0$, so that the weight 0 sub algebra of $A(D_c)$ is given by $t = 1 = r$. So the cohomology of $A(D_c)$ is the Eilenberg Maclane

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cohomology of the weight 0 generating space of D_c is given by the conditions $t = 0 = r$ in D_c and is linearly spanned by the derivations $x_i \partial x_j$ with $t_i = t_j$.

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9.3 Examples

Example 9.1. *Consider $\mathcal{F} = \mathbb{C}P^n$ and $F = S(x, y)$ with $|x| = 2, |y| = 2n + 1$ and $dy = x^{n+1}$. Since F is free and finitely generated, we take $D = Der\ F$ and obtain*

$$D = \{\theta^0, \theta^{-2}, \phi^{-1}, \phi^{-3}, \dots, \phi^{-2n-1}\}$$

with indexing denoting degree and

$$\begin{aligned} \theta^0 &= 2x\partial x + (2n+2)y\partial y \\ \theta^{-2} &= \partial x \\ \phi^{-(2k+1)} &= x^{n-k}\partial y. \end{aligned}$$

The only nonzero differential is $d\theta^{-2} = \phi^{-1}$ and the nonzero brackets are

$$\begin{aligned} [\theta^0, \theta^{-2}] &= 2\theta^{-2} \\ [\theta^0, \phi^\nu] &= (\nu - 1)\phi^\nu \quad \text{and} \\ [\theta^{-2}, \phi^\nu] &= (n-k)\phi^{\nu-2}. \end{aligned}$$

We then have the sub-dg Lie algebra $D_c = \{\theta^0, \phi^{-3}, \dots\}$ which yields

$$A(D_c) \simeq S(v^1, w^4, w^6, \dots, w^{2n+2}),$$

the free algebra with $dv^1 = 0, dw^4 = \frac{1}{2}v^1 w^4$, etc., and $v^1, w^4, \dots$ dual to $\theta^0, \phi^{-3}, \dots$. The cohomology of this dgca is that of the subalgebra $S(v^1)$ (by the theorem above), which here is a model of $BG = K(G, 1)$ for $G = GL(1)$, the (discrete) group of homotopy classes of homotopy equivalences of $\mathbb{C}P^n$. (These automorphisms are represented geometrically by the endomorphisms $(z, \dots, z_n) \mapsto (z^\lambda, \dots, z_n^\lambda)$ of the rationalization of $\mathbb{C}P^n$. this formula is not well defined on $\mathbb{C}P^n$ unless q is an integer, but does extend to the rationalization for all q in $\mathbf{Q}^*$. This follows from general principles, but may also be seen explicitly as follows. The rationalization is the inverse limit of $\mathbb{C}P^n_s$, indexed by the positive integers s, which are ordered by divisibility. The transition maps $\mathbb{C}P^n_s \to \mathbb{C}P^n_t$ are given by $z_i \mapsto z_i^{s/t}$. The m-th root of the sequence (x_s) is then (y_s), where $y_s = x_{ms}$.

Algebraically these grading automorphisms of the formal dga F are given by $a \mapsto t^w a$ (w = weight of a) for a in F.

Thus, the characteristic classes in $HA(D_c)$ have detected only the fibrations over S^1, not the remaining fibrations given by

$$[A(D_c), S^{2k}] = H^{-2k-1}(D) = \{[\phi^{-2k-1}]\}$$

"dual" to w^{2k}.

These other fibrations are, however, detected by

$$H(A(D_s)) = S(w^4, w^6, \ldots, w^{2n+2}),$$

since $D_s = \{\theta^{-2}, \phi^{-1}, \phi^{-3}, \ldots\}$ has the homotopy type of $\{\phi^{-3}, \ldots\}$. These last fibrations come from standard $\mathbb{C}^{n+1}$ vector bundles over S^{2k}, and the w^{2i} correspond to Chern classes c_i via the map $BGL(n+1, \mathbb{C}) \to BS\,Aut\,\mathbb{C}P^n$. (The fibration for c_1 is missing because, for $n = 0$, the map $BGL(1, \mathbb{C}) \to BS\,Aut^* \simeq *$ is trivial; a projectivized line bundle is trivial.

To look at some other examples, we use computational machinery and the notation of Section 8.

If the positive homology of F is spanned by $x_1, \ldots, x_r$ of degrees $\nu_1, \ldots, \nu_r, r > 1$, then $Der\,L(\mathcal{H})$ is spanned by symbols of the form

$$[x_1, [x_2, [\ldots, x_{m+1}] \ldots] \partial x_{m+2}$$

of degree $\nu_{m+2} - (\nu_1 + \cdots + \nu_{m+1}) + m$.

Example 9.2. *If we take the fibre to be the bouquet $S^\nu \vee \cdots \vee S^\nu$ (r times), then $\nu_i = \nu, d = 0$ in LH and in D, and the weight 0 part of D_c has weight is*

$$\mathfrak{g} = \{x_i \partial x_j\} \simeq \mathfrak{gl}(r)$$

and

$$H(A(D_c)) = H(A(g)) = S(v^1, v^3, \ldots, v^{2n-1})$$

(superscripts again indicate degrees) and detects over S^1 the fibrations (with fibre the bouquet) which are obtained by twisting with an element of $GL(r)$. The model $A(D_c)$ has homotopy groups in degree $p = m(\nu - 1) + 1$, spanned by symbols as above (mod $ad\,L(\mathcal{H})$). Such a homotopy group is generated by a map corresponding to a fibration over S^p with twisting term

$$\tau \in H^p(S^p) \otimes H^{1-p}(Der\ F)$$

which has weight $1 - m$ in $H(S^p \otimes F)$. For $m > 1$, this is negative and gives a perturbation of the homotopy type of $S^p \times F$ (fixing the cohomology); we thus have for $m > 1$, a surjection from fibration classes $F \to E \to S^p$ to homotopy types with cohomology $H(S^p \otimes F)$, the kernel being given by the orbits of $GL(r)$ acting on the set of fibration classes. For $m = 1, p = \nu$, the twisting term gives a new graded algebra structure to $H(S^p \otimes F)$ via the structure constants a_{ij}^k which give $x_i x_j = \Sigma a_{ij}^k y x_k$ where y generates $H^p(S^p)$.

If we replace the base S^ν by $K(\mathbf{Q},\nu)$ (for $\nu = 2, \mathbb{C}P^2$ will suffice), then the integrability condition $[\tau,\tau] = 0$ is no longer automatic; it corresponds to the associativity condition on the r dimensional vector space $H^\nu(F)$ with multiplication given by structure constants a^k_{ij}. The cohomology of $B\,S\,Aut\,(S^\nu \vee \cdots \vee S^\nu)$ generated by degree ν is the coordinate ring of the (miniversal) variety of associative commutative unitary algebras of dimension $r+1$; that is, it is isomorphic to the polynomial ring on the symbols a^k_{ij} modulo the quadratic polynomials expressing the associativity condition, and the r linear polynomials arising from the action of $ad\,x_i$ (“translation of coordinates”).

Apart from these low degree generators and relations, the cohomology of $A(D_s)$ remains to be determined. For example, is it finitely generated as an algebra? Already for the case of $S^2 \vee S^2$, there is, beside the above classes in $H^2(A(D_s))$, an additional generator in H^3 dual to

$$\theta = [x_1,[x_1,x_2]]\partial x_1 - [x_2[x_1,x_2]]\partial x_2 \in D^{-2}$$

(which gives the nontrivial fibration $S^2 \vee S^2 \to E \to S^3$ considered before). Since $\theta \in [D_s, D_s]$, it yields a nonzero cohomology class.

In this last example, we saw that $H^*(E)$ need not be $H^*(B) \otimes H^*(F)$ as an algebra. Included in our classification are fibrations in which $H(E)$ is not even additively isomorphic to $H(B) \otimes H(F)$.

Example 9.8. *Consider the case in which the fibre is S^ν. For ν odd, we have $F = S(x)$ and Der $F = S(x)\partial x$ with $D_s = \{\partial x\}$. The universal simply connected fibration is*

$$S^\nu \to E \to K(\mathbf{Q},\nu+1)$$

or

$$S(x) \leftarrow (S(x,u), dx = u) \leftarrow S(u)$$

with E contractible. Here $\tau = \partial x \otimes u$ and the transgression is not zero.

By contrast, when ν is even, $F = S(x,y)$ with $dy = x^2$ and we get $D_s = \{y\partial x, \partial x, \partial y\}$ which is homotopy equivalent to $\{\partial y\}$. The universal simply connected S^2–fibration is then

$$S^\nu \to E \to K(\mathbf{Q},2\nu)$$

where $E = (S(x,y,u), dy = x^2 - u) \simeq S(x)$ is the model for $K(\mathbf{Q},\nu)$. Here $\tau = \partial y \otimes u$ gives a “deformation” of the algebra $H(B) \otimes H(F) = S(x,u)/x^2$ to the algebra $H(E) = S(x,u)/(x^2-u) = S(x)$.

(Fibrations

$$\bigvee_1^n S^{2(n+i)} \to E \to K(\mathbf{Q},2(n+2))$$

occur in Tanré’s analysis [66] of homotopy types related to the stunted infinite dimensional complex projective spaces $\mathbb{C}P^\infty_n$.)

A neat way to keep track of these distinctions is to consider the Eilenberg–Moore filtration of $B \otimes F$ where F is a filtered model, i.e., weight $(b \otimes f) =$ degree b + weight $f =$ deg b + deg f + resolution degree f. (Cf. [71].)

In general, $\tau \in (Der\ F \hat{\otimes} B)1$ will have weight ≤ 1 since weight $f \leq$ degree f and τ does not increase F degree. If, in fact, weight $\tau \leq 0$, then $H(E)$ is isomorphic to $H(B) \otimes H(F)$ as $H(B)$–module but not necessarily as $H(B)$–algebra.

Finally, if we can accept dgca's with negative degrees (without truncating so as to model a space), we can obtain a uniform description of fibrations and perturbations of the homotopy type F. Consider in $Der\, F$, the sub-dg Lie algebra D_- of negatively weighted derivations, then $[A(D_-), B]$ is for $B = S^0$, the space of homotopy types underlying $H(F)$ while for connected B, it is the space of strong equivalence classes of F–fibrations over B.

9.4 Open questions

We turn now to the question of realizing a given quotient variety $M = V/G$ as the set of fibrations with given fibre and base, or as the set of homotopy types with given cohomology. The structure of M appears to be arbitrary, except that V must be conical (and for fibrations, G must be pro-unipotent). The fibrations of an odd dimensional sphere S^ν over B form an affine space $M = V = H^{\nu+1}(B)$, though it is not clear how to make V have general singularities or pro-unipotent group action.

For homotopy types, we consider the following example, provoked by a letter from Clarence Wilkerson. Take F to be the model of $S^\nu \vee S^\nu$, for ν even. As we have seen, the model D of $Der\, F$ contains a derivation $\theta = [x_1, [x_1, x_2]]\partial x_1 - [x_2, [x_1, x_2]]\partial x_2$ of degree $-2\nu + 2$ and weight -2ν, which generates $H^{-2\nu+2}(D)$. If $B = S^3 \times (CP^\infty)^n$, then $H^{-2\nu+2}(D) \otimes H^{2\nu-1}(B) = V$ has weight -1 and may be identified with the homogeneous polynomials of degree $\nu - 2$ in r variables. If we truncate B suitably, so that we have $H^1((D \otimes B)_-) = V$, then the set of rational homotopy types with cohomology equal to $H(F) \otimes H(B)$ is the quotient $V/GL(r)$, i.e., equivalence classes of polynomials of (even) degree $\nu = 2$ in r variables.

We may ask, similarly, which dgca's occur, up to homotopy types, as classifying algebras $A(D_c)$ or $A(D_s)$. The general form of the representation problem is the following: given a finite type of dgL D, does there exist a free dg Lie algebra π such that $D \sim Der\ \pi/ad\,\pi$?

10 Postscript

Some n years after a preliminary version of this preprint first circulated, there have been major developments of the general theory and significant applications, many inspried by the interaction with physics. We have not tried to describe them; a book would be more appropriate to address properly this active and rapidly evolving field.

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